Adjoint Functors in Algebra, Topology and Mathematical Logic

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1 Introduction

In mathematics we enjoy different ways of comparing objects to each other. We usually use isomorphisms to say two objects are the same, but this is a very strong condition. In category theory we have a weaker condition for saying two categories are related known as equivalence, but there is an even weaker condition known as adjunction. This paper explores different definitions of adjoint functors and there equivalences, as well as examples of adjoint functors in algebra, topology and mathematical logic.

2 Relating Categories

When comparing objects in a category we use morphisms, similarly we need a way to have morphisms between categories. Functors are the morphisms we use.

Definition 2.1 (Functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *covariant functor* $F : \mathcal{C} \to \mathcal{D}$ assigns each $\mathcal{C}$-object X a $\mathcal{D}$-object $F(X)$ and each $\mathcal{C}$-morphism $f : A \to B$ a $\mathcal{D}$-morphism $F(f)$, such that

- $F(1_X) = 1_{F(X)}$,
- $F(g \circ f) = F(g) \circ F(f)$ whenever $g \circ f$ is defined

for all $\mathcal{C}$-objects X and $\mathcal{C}$-morphisms f, g. [Lan98, p.13].

It is easy to show that functors can be composed (using the obvious composition) and that composition is associative. There is also the *identity functor* $\mathbf{I}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ which maps each object and morphism to itself.

Using functors and the identity functor we can now define what it means for two categories to be isomorphic:

Definition 2.2 (Isomorphic Categories)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. $\mathcal{C}$ and $\mathcal{D}$ are *isomorphic* if there exists functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that

$$\begin{aligned} \mathbf{I}_{\mathcal{C}} &= GF \\ FG &= \mathbf{I}_{\mathcal{D}} \end{aligned}$$

[Lan98, p.14]

We can also define a weaker condition known as equivalence:

Definition 2.3 (Equivalent Categories)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there exists functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that

$$\begin{aligned} \mathbf{I}_{\mathcal{C}} &\cong GF \\ FG &\cong \mathbf{I}_{\mathcal{D}} \end{aligned}$$

[Lan98, p.18]

So instead of requiring $GF(X) = X$ and $FG(Y) = Y$, we instead require that $GF(X)$ is naturally isomorphic to X and $FG(Y)$ is naturally isomorphic to Y. So F need not be a bijection. Naturally isomorphic means there exists a natural isomorphism. This means that each component in the natural transformation is an isomorphism (we will define natural transformations soon). Like most isomorphisms, natural isomorphisms preserve the structure of the functors.

There is an obvious similarity between the definition of isomorphic categories and equivalent categories. To define adjunction we use natural transformations instead of equality or natural isomorphisms between functors.

Definition 2.4 (Natural Transformation)**.** Given 2 functors $F, G : \mathcal{C} \to \mathcal{D}$, a *natural transformation* $\eta : F \dot{\to} G$ assigns each $\mathcal{C}$-object X a $\mathcal{D}$-morphism $\eta_X : F(X) \to G(X)$ such that for any $\mathcal{C}$-morphism $f : X \to Y$ we get that

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"] & G(Y) \end{tikzcd}$$

commutes [Lan98, p.16].

Natural transformations can be thought of as morphisms between functors. Each η_X is called the *component* of η at X. If every component is isomorphic, then η is said to be a *natural isomorphism*.

Using the pattern above the definition of adjunction is there exists natural transformations $\eta : \mathbf{I}_{\mathcal{C}} \dot{\to} GF$ and $\epsilon : FG \dot{\to} \mathbf{I}_{\mathcal{D}}$. Just requiring 2 transformations is a very weak condition, so we additionally need conditions on the natural transformations. The condition turns out to be that each component η_X is a universal morphism to G from X.

Definition 2.5 (Universal Morphism)**.** If $F : \mathcal{D} \to \mathcal{C}$ is a functor and X an object of $\mathcal{C}$, a *universal morphism* X to F is a pair $\langle X', u\rangle$ consisting of a $\mathcal{D}$-object X' and a $\mathcal{C}$-morphism $u : X \to F(X')$, such that to every pair $\langle Y, f\rangle$ with $\mathcal{D}$-object Y and $\mathcal{C}$-morphism $f : X \to F(Y)$, there is a unique $\mathcal{D}$-morphism $f' : X' \to Y$ with $F(f') \circ u = f$ [Lan98, p.55].

Definition 2.6 (Adjunction)**.** An *adjunction* consists of the following:

- categories $\mathcal{C}$ and $\mathcal{D}$,
- functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$,
- natural transformation $\eta : \mathbf{I}_{\mathcal{C}} \dot{\to} GF$.

This forms an *adjunction* if $\langle FX, \eta_X\rangle$ is a universal morphism from X to G for every $\mathcal{C}$-object X. [Lan98, p.81].

F is called the *left adjoint* and G is called the *right adjoint*. We call η the *unit* of the adjunction. We will refer to the above definition as the *unit universal morphism definition*.

Since every η_X is universal, we have for every $\mathcal{C}$-morphism $f : X \to G(Y)$ a unique $\mathcal{D}$-morphism $g : F(X) \to Y$ such that

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & GF(X) \arrow[d, dashed, "G(g)"] \\ & G(Y) \end{tikzcd}$$

commutes.

We can also define adjunction using the *counit* $\epsilon : FG \dot{\to} \mathbf{I}_{\mathcal{D}}$.

Definition 2.7. Given functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ and the natural transformation $\epsilon : FG \dot{\to} \mathbf{I}_{\mathcal{D}}$, F is left adjoint to G if $\langle \epsilon_Y, GY\rangle$ is a universal morphism from F to Y for every $\mathcal{D}$-object Y [Lan98, p.81].

ϵ is called the *counit* of the adjunction. We will refer to this definition as the *counit universal morphism definition*.

Notice that the universal morphism is from a functor to an object. This is just the dual of the universal morphism definition. So for every $\mathcal{C}$-morphism $f : X \to G(Y)$ there is a unique $\mathcal{D}$-morphism $g : F(X) \to Y$ such that

$$\begin{tikzcd} F(X) \arrow[d, "F(f)"'] \arrow[dr, dashed, "g"] & \\ FG(Y) \arrow[r, "\epsilon_Y"] & Y \end{tikzcd}$$

commutes.

There is a definition for adjunction using the unit and the counit without their universal morphism properties.

Definition 2.8. $\langle F, G, \eta, \epsilon \rangle : \mathcal{C} \to \mathcal{D}$ where

- $\mathcal{C}$ and $\mathcal{D}$ are categories,
- $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are functors,
- $\eta : \mathbf{I}_{\mathcal{C}} \dot{\to} GF$ and $\epsilon : FG \dot{\to} \mathbf{I}_{\mathcal{D}}$ are natural transformation

is an *adjunction* if

$$\begin{aligned} 1_{FX} &= \epsilon_{FX} \circ F(\eta_X) \\ 1_{GY} &= G(\epsilon_Y) \circ \eta_{GY} \end{aligned}$$

for each $\mathcal{C}$-object X and $\mathcal{D}$-object Y [Lan98, p.81].

We will refer to this definition as the *unit and counit definition.*

3 Adjunction in terms of hom-sets

The above definitions are useful for getting an intuitive understanding of adjoint functors. But most practicing category theorists like to think of an adjunction as a natural isomorphism between hom-sets.

$$\mathcal{D}(F(X), Y) \cong \mathcal{C}(X, G(Y))$$

We will need to build up some tools to understand this definition.

Definition 3.1 (Hom-Sets)**.** For $\mathcal{C}$-objects A and B, the *hom-set*

$$\hom_{\mathcal{C}}(A, B) = \{f | f : A \to B \text{ a } \mathcal{C}\text{-morphism}\}$$

[Lan98, p.27]

The notation $\hom_{\mathcal{C}}(A, B)$ is frequently abbreviated as

$$\hom_{\mathcal{C}}(A, B) = \mathcal{C}(A, B) = \hom(A, B) = (A, B) = (A, B)_{\mathcal{C}}$$

We will use $\mathcal{C}(A, B)$ in proofs for brevity.

So $\hom_{\mathcal{C}}(A, B)$ is the collection of all morphisms between A and B.

A common used notation for defining functors and morphisms is the use of $-$. $-$ acts as a placeholder for the input. For example, in arithmetic we can define the function $- \times 5$, which is the same as the function $f(x) = x \times 5$. Using hom-sets we can define hom-functors.

Definition 3.2 (Covariant Hom-Functor)**.** $\hom(X, -) : \mathcal{C} \to \mathbf{Set}$ is the *covariant hom-functor* where

- each $\mathcal{C}$-object Y is mapped to $\hom_{\mathcal{C}}(X, Y)$,
- each $\mathcal{C}$-morphism $f : A \to B$ is mapped to $\hom_{\mathcal{C}}(X, f) : \hom_{\mathcal{C}}(X, A) \to \hom_{\mathcal{C}}(X, B)$ where $\hom_{\mathcal{C}}(X, f)(g : X \to A) = f \circ g$.

[Lan98, p.34]

Similarly we can define $\hom(-, Y)$, but for composition to work we need the input morphism to be reversed (domain and codomain are swapped). A category defined by reversing the arrows of an existing category is known as its *opposite category*.

Definition 3.3 (Opposite Category)**.** Every category $\mathcal{C}$ has an associated *opposite category* $\mathcal{C}^{\mathrm{op}}$. The objects of $\mathcal{C}^{\mathrm{op}}$ are the objects of $\mathcal{C}$. The morphisms of $\mathcal{C}^{\mathrm{op}}$ are in a one-to-one correspondence with the morphisms in $\mathcal{C}$. The $\mathcal{C}$-morphism $f : X \to Y$ maps to the $\mathcal{C}^{\mathrm{op}}$-morphism $f^{\mathrm{op}} : Y \to X$ (The domain and codomain swap for each morphism). The composite $f^{\mathrm{op}} \circ g^{\mathrm{op}} = (f \circ g)^{\mathrm{op}}$ is defined whenever $g \circ f$ is defined in $\mathcal{C}$ [Lan98, p.33].

Definition 3.4 (Contravariant Functor)**.** $F : \mathcal{C} \to \mathcal{D}$ is a *contravariant functor* if $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is a covariant functor. So F maps $\mathcal{C}$-morphisms to there opposites [Lan98, p.33].

Since we reverse the input arrows for $\hom(-, Y)$, it is a contravariant functor with a similiar definition to $\hom(X, -)$.

Definition 3.5 (Contravariant Hom-Functor)**.** $\hom(-, Y) : \mathcal{C} \to \mathbf{Set}$ is the *contravariant hom-functor* where

- each $\mathcal{C}$-object X is mapped to $\hom_{\mathcal{C}}(X, Y)$,
- each $\mathcal{C}$-morphism $f : A \to B$ is mapped to $\hom_{\mathcal{C}}(f, Y) : \hom_{\mathcal{C}}(A, Y) \to \hom_{\mathcal{C}}(B, Y)$ where $\hom_{\mathcal{C}}(f, Y)(g : A \to Y) = g \circ f^{\mathrm{op}}$.

[Lan98, p.34]

We finally have enough tools to define adjunction in terms of hom-sets.

Definition 3.6 (Adjunction in terms of hom-sets)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. An *adjunction* from $\mathcal{C}$ to $\mathcal{D}$ is a triple $\langle F, G, \phi \rangle$ where $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are functors, while ϕ is a function which assigns to each pair of objects $X \in \mathcal{C}$, $Y \in \mathcal{D}$ a bijection

$$\phi = \phi_{X,Y} : \hom_{\mathcal{D}}(F(X), Y) \cong \hom_{\mathcal{C}}(X, G(Y))$$

which is natural in X and Y [Lan98, p.78].

Note that ϕ is natural in 2 variables. So $\hom_{\mathcal{D}}(F(X), Y)$ is actually the *bifunctor* $\hom_{\mathcal{D}}(F(-), -) : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Set}$. This just means that each pair of (X, Y) objects is sent to $\hom_{\mathcal{D}}(FX, Y)$. Similarly $\hom_{\mathcal{C}}(-, G(-))$ sends each

(X', Y') to $\hom_{\mathcal{C}}(X', G(Y'))$. The naturality is in 2 variables, so we get "2 naturality squares" for all $f : X' \to X$ and $g : Y \to Y'$:

$$\begin{tikzcd}
\mathcal{D}(F(X), Y) \arrow[r, "\phi_{X,Y}"] \arrow[d, "\hom_{\mathcal{D}}(F(f),Y)"] & \mathcal{C}(X, G(Y)) \arrow[d, "\hom_{\mathcal{C}}(f,G(Y))"] \\
\mathcal{D}(F(X'), Y) \arrow[r, "\phi_{X',Y}"] & \mathcal{C}(X', G(Y))
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{D}(F(X), Y) \arrow[r, "\phi_{X,Y}"] \arrow[d, "\hom_{\mathcal{D}}(F(X),g)"] & \mathcal{C}(X, G(Y)) \arrow[d, "\hom_{\mathcal{C}}(X,G(g))"] \\
\mathcal{D}(F(X), Y') \arrow[r, "\phi_{X,Y'}"] & \mathcal{C}(X, G(Y'))
\end{tikzcd}$$

4 Equivalence of definitions

We shall prove the equivalence of the above definitions. We will use the notation $GFX = (G \circ F)(X) = G(F(X))$, where G and F are composible functors and X is an object. Similarly we will use the notation $GFf = G(F(f))$, where f is a morphism.

First we show that the *unit universal morphism definition* implies the *hom-set definition*:

Proof. Let $F : \mathcal{C} \to D$ be left adjoint to $G : \mathcal{D} \to \mathcal{C}$ with unit η. For every $f : X \to GY$ there is a unique $g : FX \to Y$ such that

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & GFX \arrow[d, dashed, "Gg"] \\
& GY
\end{tikzcd}$$

commutes.

So $\phi(g) = Gg \circ \eta_X$ defines a bijection

$$\phi : \mathcal{D}(FX, Y) \to \mathcal{C}(X, GY)$$

ϕ is natural in X since η is natural, and natural in Y since G is a functor. So $\langle F, G, \phi \rangle$ gives an adjunction. $\square$

Now we show that the *counit universal morphism definition* implies the *hom-set definition*:

Proof. Let $F : \mathcal{C} \to D$ be left adjoint to $G : \mathcal{D} \to \mathcal{C}$ with counit ϵ. For every $f : X \to GY$ there is a unique morphism $g : FX \to Y$ such that

$$\begin{tikzcd}
FX \arrow[d, "Ff"'] \arrow[dr, dashed, "g"] & \\
FGY \arrow[r, "\epsilon_Y"] & Y
\end{tikzcd}$$

commutes.

So $\phi^{-1}(f) = \epsilon_Y \circ Ff$ defines a bijection

$$\phi^{-1} : \mathcal{C}(X, GY) \to \mathcal{D}(FX, Y)$$

ϕ^{-1} is natural in Y since ϵ is natural, and natural in X since F is a functor. So $\langle F, G, \phi \rangle$ gives an adjunction. $\square$

Now we show that the *hom-set definition* implies the *universal morphism definitions* for the *unit* and *counit*.

Proof. Let $\langle F, G, \phi \rangle : \mathcal{C} \to \mathcal{D}$ be an adjunction. $\phi = \phi_{X,Y} : \hom_{\mathcal{D}}(FX, Y) \to \hom_{\mathcal{C}}(X, GY)$ defines two "naturality squares" for all $f : X' \to X$ and $g : Y \to Y'$:

$$\begin{tikzcd}
\mathcal{D}(FX,Y) \arrow[r, "\phi"] \arrow[d, "-\circ Ff"] & \mathcal{C}(X,GY) \arrow[d, "-\circ f"] \\
\mathcal{D}(FX',Y) \arrow[r, "\phi"] & \mathcal{C}(X',GY)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{D}(FX,Y) \arrow[r, "\phi"] \arrow[d, "g\circ -"] & \mathcal{C}(X,GY) \arrow[d, "Gg\circ -"] \\
\mathcal{D}(FX,Y') \arrow[r, "\phi"] & \mathcal{C}(X,GY')
\end{tikzcd}$$

Note that we leave out the subscript for ϕ because it is obvious which bijection we are talking about from the context.

Setting $Y = FX$ we define η_X by seeing where $\phi_{X,FX}$ maps the identity morphism 1_{FX}.

$$\eta_X = \phi(1_{FX})$$

Also from the left diagram we get the equation

$$\phi(Ff) = \eta_X \circ f \tag{1}$$

and from the right diagram we get the equation

$$\phi(g) = Gg \circ \eta_X \tag{2}$$

Similarly for ϵ we set $X = GY$ and define

$$\epsilon_Y = \phi^{-1}(1_{GY})$$

We also get from the left diagram the equation

$$\phi^{-1}(f) = \epsilon_Y \circ Ff \tag{3}$$

and from the right diagram we get the equation

$$\phi^{-1}(Gg) = g \circ \epsilon_Y \tag{4}$$

We need to show that $\eta : \mathbf{I}_\mathcal{C} \dot{\to} GF$ and $\epsilon : FG \dot{\to} \mathbf{I}_\mathcal{D}$ are natural transformations and that η and ϵ satisfy the universal morphisms for the unit and counit respectively.

We claim that the naturality diagram for η

$$\begin{tikzcd} X' \arrow[r, "\eta_{X'}"] \arrow[d, "f"] & GFX' \arrow[d, "GFf"] \\ X \arrow[r, "\eta_X"] & GFX \end{tikzcd}$$

commutes.

Now

$$\begin{aligned} \eta_X \circ f &= \phi(Ff) && \text{Using (1)} \\ &= GFf \circ \eta_{X'} && \text{Using (2) with } g = Ff \end{aligned}$$

Hence the diagram commutes, so η is a natural transformation.

Similarly we claim that the naturality diagram for ϵ

$$\begin{tikzcd} FGY \arrow[r, "\epsilon_Y"] \arrow[d, "FGg"] & Y \arrow[d, "g"] \\ FGY' \arrow[r, "\epsilon_{Y'}"] & Y' \end{tikzcd}$$

commutes.

Now

$$\begin{aligned} \epsilon_{Y'} \circ FGg &= \phi^{-1}(Gg) && \text{Using (3) with } f = Gg \\ &= g \circ \epsilon_Y && \text{Using (4)} \end{aligned}$$

Hence the diagram commutes, so ϵ is a natural transformation.

We also claim that η satisfies the unit universal morphism axioms. Let $f : X \to GY$. We need to find a unique morphism $g : FX \to Y$ such that

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & GFX \arrow[d, dashed, "Gg"] \\ & GY \end{tikzcd}$$

commutes.

Let $g = \phi^{-1}(f)$. Using (2) we get

$$f = \phi(\phi^{-1}(f)) = \phi(g) = Gg \circ \eta_X$$

So the diagram commutes. Since ϕ is a bijection, g is unique in making the diagram commute. Hence $\langle FX, \eta_X \rangle$ is a universal morphism to G from X for every $\mathcal{C}$-object X.

We also claim that ϵ satisfies the counit universal morphism axioms. Let $f : X \to GY$. We need to find a unique morphism $g : FX \to Y$ such that

$$\begin{tikzcd} FX \arrow[d, "Ff"'] \arrow[dr, dashed, "g"] & \\ FGY \arrow[r, "\epsilon_Y"] & Y \end{tikzcd}$$

commutes.

Let $g = \phi^{-1}(f)$. Using (3) we get

$$g = \phi^{-1}(f) = \epsilon_Y \circ Ff$$

So the diagram commutes. Since ϕ is a bijection, g is unique in making the diagram commute. Hence $\langle \epsilon_Y, GY \rangle$ is a universal morphism from F to Y for every $\mathcal{D}$-object Y. □

Finally we will show that the *hom-set definition* is equivalent to the *unit and counit definition.* First we show that hom-set implies unit and counit.

Proof. We use the same definition for η and ϵ from the previous proof.

$$\begin{aligned} \eta_X &= \phi(1_{FX}) \\ \epsilon_Y &= \phi^{-1}(1_{GY}) \end{aligned}$$

We need to show $1_{FX} = \epsilon_{FX} \circ F(\eta_X)$ and $1_{GY} = G(\epsilon_Y) \circ \eta_{GY}$. The naturality square induced from $\phi_{GFX,FX}$ with $\eta_X : X \to GFX$ is

$$\begin{tikzcd} \mathcal{D}(FGFX, FX) \arrow[r, "\phi"] \arrow[d, "- \circ F\eta_X"] & \mathcal{C}(GFX, GFX) \arrow[d, "- \circ \eta_X"] \\ \mathcal{D}(FX, FX) & \mathcal{C}(X, GFX) \arrow[l, "\phi^{-1}"] \end{tikzcd}$$

Now $\epsilon_{FX} : FGFX \to FX$ is an element of $\mathcal{D}(FGFX, FX)$. Using the above diagram we get

$$\epsilon_{FX} \circ F\eta_X = \phi^{-1}(\phi(\epsilon_{FX}) \circ \eta_X) = \phi^{-1}(1_{GFX} \circ \eta_X) = \phi^{-1}(\eta_X) = 1_{FX}$$

Similarly we use the naturality square induced from $\phi_{FGY,FGY}$ with $\epsilon_Y : FGY \to Y$.

$$\begin{tikzcd} \mathcal{D}(FGY, FGY) \arrow[d, "\epsilon_Y \circ -"] & \mathcal{C}(GY, GFGY) \arrow[l, "\phi^{-1}"] \arrow[d, "G\epsilon_Y \circ -"] \\ \mathcal{D}(FGY, Y) \arrow[r, "\phi"] & \mathcal{C}(GY, GY) \end{tikzcd}$$

Now since $\eta_{GY} : GY \to GFGY$ is an element of $\mathcal{C}(GY, GFGY)$ we get

$$G\epsilon_Y \circ \eta_{GY} = \phi(\epsilon_Y \circ \phi^{-1}(\eta_{GY})) = \phi(\epsilon_Y \circ 1_{GY}) = \phi(\epsilon_Y) = 1_{GY}$$

□

Now we show that unit and counit definition imply hom-set definition.

Proof. We use η and ϵ to define functions $\phi : \mathcal{D}(FX, Y) \to \mathcal{C}(X, GY)$ and $\sigma : \mathcal{C}(X, GY) \to \mathcal{D}(FX, Y)$.

$$\begin{aligned} \phi(f : FX \to Y) &= Gf \circ \eta_X \\ \sigma(g : X \to GY) &= \epsilon_Y \circ Fg \end{aligned}$$

Now

$$\begin{aligned} \phi(\sigma(g)) &= G(\epsilon_Y \circ Fg) \circ \eta_X \\ &= G\epsilon_Y \circ GFg \circ \eta_X && (G \text{ is a functor}) \\ &= G\epsilon_Y \circ \eta_{GY} \circ g && (\text{Naturality of } \eta) \\ &= 1_{GY} \circ g && (\text{By hypothesis } G\epsilon_Y \circ \eta_{GY} = 1_{GY}) \\ &= g \end{aligned}$$

Hence $\phi\sigma$ is the identity. Similarly $\sigma\phi$ is the identity. Therefore ϕ is a bijection with inverse σ. It is clearly natural from its definition, hence forms an adjunction. $\square$

5 Summary of adjunction

We now have a few different definitions for adjunction. Here is a summary for an adjunction between $\mathcal{C}$ and $\mathcal{D}$

- A functor $F : \mathcal{C} \to \mathcal{D}$ called the *left adjoint*
- A functor $G : \mathcal{D} \to \mathcal{C}$ called the *right adjoint*
- A natural transformation $\eta : \mathbf{I}_{\mathcal{C}} \dot{\to} GF$ called the *unit*
- A natural transformation $\epsilon : FG \dot{\to} \mathbf{I}_{\mathcal{D}}$ called the *counit*
- A natural isomorphism $\phi = \phi_{X,Y} : \hom_{\mathcal{D}}(FX, Y) \to \hom_{\mathcal{C}}(X, GY)$.

We also discovered a few important equations in our proofs that relate the different concepts of an adjunction. Here is a brief list of the equations:

$$\begin{aligned} \eta_X &= \phi_{X,FX}(1_{FX}) \\ \epsilon_Y &= \phi^{-1}_{GY,Y}(1_{GY}) \\ 1_{FX} &= \epsilon_{FX} \circ F\eta_X \\ 1_{GY} &= G\epsilon_Y \circ \eta_{GY} \\ g &= \phi^{-1}_{X,Y}(f) = \epsilon_Y \circ Ff \\ f &= \phi_{X,Y}(g) = Gg \circ \eta_X \end{aligned}$$

where $g : FX \to Y$ and $f : X \to GY$ are the morphisms from the universal morphism definitions.

6 Examples from Algebra

Adjoint functors are very common in algebra, especially when universal constructions are involved. One of the main universal constructions we did was that of free algebras.

Definition 6.1 (Free Algebra)**.** For a full subcategory $\mathcal{C}$ in Ω-**Alg** and a set X, the free algebra $F_{\mathcal{C}}(X)$ in $\mathcal{C}$ over X is defined as an initial object $F_{\mathcal{C}}(X) = (F_{\mathcal{C}}(X), \psi)$ in the category $\mathcal{C}[X]$ [Jan08, p.16].

We will often write $F(X)$ instead of $F_{\mathcal{C}}(X)$. So picking a $\mathcal{C}$, a set X and a map $\mu : X \to F(X)$, we get that for every $\mathcal{C}$-object B and map $f : X \to B$ we get a unique $\mathcal{C}$-morphism $g : F(X) \to B$ such that

$$\begin{tikzcd} F(X) \arrow[dr, dashed, "g"] & \\ X \arrow[u, "\mu"] \arrow[r, "f"'] & B \end{tikzcd}$$

commutes. This diagram looks a lot like the diagram for universal morphisms using the unit. F can be defined as the functor $F : \mathbf{Set} \to \mathcal{C}$ which maps each set to its freely generated object. F is the left adjoint. What we need now is a right adjoint. In this case it is the forgetful functor $U : \mathcal{C} \to \mathbf{Set}$ which assigns each object to its underlying set. All that is left is to relabel μ to η_X. We now have the diagram

$$\begin{tikzcd} UFX \arrow[dr, dashed, "Ug"] & \\ X \arrow[u, "\eta_X"] \arrow[r, "f"'] & UB \end{tikzcd}$$

We will look at the *free monoids* generated by X. We define $F(X)$ as $(X^*, +, \langle\rangle)$, where X^* is the set of all finite lists generated from X, $+$ is list concatenation and $\langle\rangle$ is the empty list. For example if $X = \{a, b\}$ then

$$X^* = \{\langle a\rangle, \langle b\rangle, \langle a, a\rangle, \langle a, b\rangle, \langle b, a\rangle, \langle b, b\rangle, \langle a, a, a\rangle, ...\}$$

and $\langle a, b, a\rangle + \langle b, b\rangle = \langle a, b, a, b, b\rangle$. $F(X)$ is obviously a monoid.

Proposition 6.1. *F : **Set** $\to$ **Mon** defined above is left adjoint to the forgetful functor U : **Mon** $\to$ **Set** with the unit η defined by $\eta_X(x) = \langle x\rangle$ for all $x \in X$.*

Proof. Let $B = (UB, *, 1)$ be a **Mon**-object and $f : X \to UB$ be a **Set**-morphism. We need to find a unique $g : FX \to B$ such that $Ug \circ \eta_X = f$. We define g as follows

$$g(\langle s_1, s_2, ..., s_n\rangle) = f(s_1) * f(s_2) * ... * f(s_n)$$

g is a monoid homomorphism since

$$\begin{aligned} g(\langle s_1, ..., s_n\rangle + \langle t_1, ..., t_m\rangle) &= g(\langle s_1, ..., s_n, t_1, ..., t_m\rangle) \\ &= f(s_1) * ... * f(s_n) * f(t_1) * ... * f(t_m) \\ &= (f(s_1) * ... * f(s_n)) * (f(t_1) * ... * f(t_m)) \\ &= g(\langle s_1, ..., s_n\rangle) * g(\langle t_1, ..., t_m\rangle) \end{aligned}$$

and $g(\langle\rangle) = 1$.

$$Ug(\eta_X(x)) = Ug(\langle x\rangle) = f(x)$$

hence the diagram commutes.

Now let g' be a monoid homomorphism which makes the diagram commute. We need to show $g = g'$. Let $\langle s_1, ..., s_n\rangle \in X^*$. By induction on n, the number of elements in the list, we show that $g(\langle s_1, ..., s_n\rangle) = g'(\langle s_1, ..., s_n\rangle)$.
Base: $n = 0$. $g(\langle\rangle) = 1 = g'(\langle\rangle)$ since g and g' are monoid homomorphisms.
Induction step: Assume $g(\langle s_1, ..., s_{n-1}\rangle) = g'(\langle s_1, ..., s_{n-1}\rangle)$.

$$\begin{aligned} g(\langle s_1, ..., s_{n-1}, s_n\rangle) &= g(\langle s_1, ..., s_{n-1}\rangle + \langle s_n\rangle) \\ &= g(\langle s_1, ..., s_{n-1}\rangle) * g(\langle s_n\rangle) && g \text{ is a monoid homomorphism} \\ &= g'(\langle s_1, ..., s_{n-1}\rangle) * g(\eta_x(s_n)) && \text{Induction hypothesis} \\ &= g'(\langle s_1, ..., s_{n-1}\rangle) * f(s_n) && \text{Commutativity of diagram} \\ &= g'(\langle s_1, ..., s_{n-1}\rangle) * g'(\eta_x(s_n)) && \text{Commutativity of diagram} \\ &= g'(\langle s_1, ..., s_{n-1}\rangle) * g'(\langle s_n\rangle) \\ &= g'(\langle s_1, ..., s_{n-1}\rangle + \langle s_n\rangle) && g' \text{ is a monoid homomorphism} \\ &= g'(\langle s_1, ..., s_{n-1}, s_n\rangle) \end{aligned}$$

Hence g is uniquely makes the diagram commute. □

An interesting application of this adjunction is that if you fix B to the monoid of the natural numbers with addition, $g : \mathbf{List}(X) \to \mathbb{N}$ is the functor *length*, which assigns to each list the number of elements in it [Pie91, p.46].

It can be shown that the free object functor is left adjoint to the forgetful functor for other familiar structures such as modules, vector spaces and groups.

We can also define some categorical concepts using adjoints. In this example we use the category **1**, which contains only one object and the identity morphism of that object.

Proposition 6.2. *An initial object* 0 *in a category* $\mathcal{C}$ *arises as the image of the unique object of the category* ***1*** *under a left adjoint to the constant functor* $T : \mathcal{C} \to \mathbf{1}$ *[Pie91, p.48].*

Proof. Let F be a functor that is left adjoint to T and ϕ the natural isomorphism between the hom-sets. Let $g : Y \to Y'$ be a $\mathcal{C}$-morphism. Then

$$\begin{tikzcd}
\mathcal{C}(F1, Y) \arrow[r, "\phi"] \arrow[d, "g\circ -"] & \{\mathrm{id}_1\} \\
\mathcal{C}(F1, Y') \arrow[ur, "\phi"'] &
\end{tikzcd}$$

commutes.

ϕ is a bijection, so $\mathcal{C}(F1, Y)$ and $\mathcal{C}(F1, Y')$ contain only one morphism. So for any object Y which has a morphism $g : Y \to Y'$, there is only 1 morphism $h : F1 \to Y$ and 1 morphism $h' : F1 \to Y'$ such that $g \circ h = h'$. So $F1$ is an initial object. $\square$

We can also define equivalence of categories using adjunctions. Given an adjunction $\langle F, G, \eta, \epsilon\rangle : \mathcal{C} \to \mathcal{D}$, it is easy to see from the definition that if η and ϵ are natural isomorphisms, $\mathcal{C}$ is equivalent to $\mathcal{D}$.

7 Examples from Topology

Definition 7.1 (Category of topological spaces)**. Top** is the category of topological spaces, where the objects are topological spaces and the morphisms are continuous maps.

Now we can look at a very interesting functor which has both a left and a right adjoint. G : **Top** $\to$ **Set** is the forgetful functor which assigns each topological space to its underlying set. F : **Set** $\to$ **Top** is the functor which assigns each set to its discrete topology. The discrete topology on X is the powerset of X (Every subset is open). F is left adjoint to G.

Proof. Let the unit be $\eta_X(x) = x$ for all x in X. For this to form an adjunction we need for each $f : X \to GY$ a unique continuous map $g : FX \to Y$ such that

$$\begin{tikzcd}
GFX \arrow[dr, dashed, "Gg"] & \\
X \arrow[u, "eta_X"] \arrow[r, "f"'] & GY
\end{tikzcd}$$

commutes.

Picking $Gg = g = f$ gives us a unique morphism making the diagram commute. g is a continuous map since every set is open in FX ($g^{-1}(B)$ is open for all open $B \subset Y$). $\square$

We also have the functor H : **Set** $\to$ **Top** which assigns each topological space its indiscrete topology. The indiscrete topology on X is just $\{\emptyset, X\}$. H is right adjoint to G.

Proof. Let the unit be $\eta_X(x) = x$ for all x in X. The only open subsets of HGX are $\emptyset$ and HGX. $\eta_X^{-1}(\emptyset) = \emptyset$ is open and $\eta_X^{-1}(HGX) = X$ is open. Hence η_X is a continuous map.

For this to form an adjunction we need for each continuous map $g : X \to HY$ a unique $h : GX \to Y$ such that

$$\begin{tikzcd} HGX \arrow[dr, dashed, "Hh"] & \\ X \arrow[u, "\eta_X"] \arrow[r, "g"'] & HY \end{tikzcd}$$

commutes.

Picking $Hh = h = g$ gives us a unique morphism making the diagram commute. The only open subsets in HY are $\emptyset$ and Y. $h^{-1}(Y) = X$ is open and $h^{-1}(\emptyset) = \emptyset$ is open. Hence Hh is a continuous map. □

Our next example involves connected components in a topology. We will denote the component x is in by $[x]$. To be able to prove the next example we will need a fact about continuous maps into indiscrete topologies.

Proposition 7.1. *$f : A \to B$ is a continuous map between the topology A and the discrete topology B if and only if*

$$f(x) = y \implies f(x') = y \text{ for all } x' \in [x]$$

We will consider the category **CTop** of topologies where the connected components are open. The functor F : **Set** $\to$ **CTop** which assigns each set to its discrete topology has a left adjoint C : **CTop** $\to$ **Set**, which assigns a topology to the set of its connected components.

Proof. Let the unit be $\eta_X(x) = [x]$ for all x in X. η_X is continuous since $\eta_X^{-1}([x]) = [x]$ is open. $f(x') = [x'] = [x]$ for all $x' \in [x]$. η is obviously natural.

For this to form an adjunction we need for each continuous map $c : X \to FY$ a unique map $f : CX \to Y$ such that

$$\begin{tikzcd} FCX \arrow[dr, dashed, "Ff"] & \\ X \arrow[u, "\eta_X"] \arrow[r, "c"'] & FY \end{tikzcd}$$

commutes.

Let $Ff([x]) = f([x]) = c(x)$ for all $x \in X$. $c(x)$ is continuous map to the discrete topology FY, so $c(x) = y \implies c(x') = y$ for all $x' \in [x]$. So f is well defined.

$$Ff(\eta_X(x)) = Ff([x]) = c(x)$$

Hence the diagram commutes. f is obviously unique from the definition. □

8 Examples from Mathematical Logic

In propositional logic there is an obvious way to construct a category. Let **Prop** be the category where all valid propositional formulas are objects. There is a morphism from X to Y if and only if X deduces Y (in symbols $X \vdash Y$). What we need now is a way to do conjunction and implication. For $(X \wedge Y)$ notice that we have $(X \wedge Y) \vdash X$ and $(X \wedge Y) \vdash Y$. Also that if $C \vdash X$ and $C \vdash Y$ we have $C \vdash (X \wedge Y)$. In the category we have constructed the morphism from C to $(X \wedge Y)$ is unique. This is in fact the definition for the product in category.

Definition 8.1 (Product)**.** The product of X and Y (if it exists) is $X \times Y$ with projection morphisms $\pi_1 : X \times Y \to X$, $\pi_2 : X \times Y \to Y$ such that for any object C with morphisms $f : C \to X$, $g : C \to Y$ there is a unique $h : C \to X \times Y$ such that

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[d, dashed, "h"] \arrow[dr, "g"] & \\ X & X \times Y \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y \end{tikzcd}$$

commutes.

So conjunction is a product in **Prop**. For implication we notice that if $X \vdash (Y \to Z)$ then by the deduction theorem $(X \wedge Y) \vdash Z$. Using modus ponens we also get $(X \wedge Y) \vdash ((Y \to Z) \wedge Y)$. This also has a categorical definition.

Definition 8.2 (Exponential object)**.** Let $\mathcal{C}$ be a category with binary products and let Y and Z be $\mathcal{C}$-objects. The exponential object Z^Y is defined by the universal morphism $\left\langle eval, Z^Y \right\rangle$ from the functor $- \times Y$ to Z.

So *eval* is modus ponens and Z^Y is $(Z \to Y)$. So if $(X \times Y) \vdash Z$ we get $X \vdash (Y \to Z)$ and

$$\begin{tikzcd} (X \times Y) \arrow[d, dashed] \arrow[dr] & \\ ((Y \to Z) \times Y) \arrow[r, "\text{eval}"'] & Z \end{tikzcd}$$

Since the definition of the exponential object uses a universal construction with the product, it is easy to show that the functor $(- \wedge Y)$ is left adjoint to $(-)^Y$.

Proof. Define the counit to be modus ponens. $(Y \to Z)$ exists for all Y and Z. From the definition of the exponential object $\langle \text{modus ponens}, (Y \to Z) \rangle$ is a universal morphism from $(- \wedge Y)$ to Y. $\square$

9 Conclusion

Many constructions in mathematics are examples of adjoints, even constructions in category theory itself. Anything that arises frequently deserves its own study, which hopefully sheds some deeper understanding on the examples they apply to. We end off this paper with a quote from Mac Lane "The slogan is, 'Adjoint functors arise everywhere' ".

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