A Propositional Typicality Logic for Extending Rational Consequence

Richard Booth, Thomas Meyer, Ivan Varzinczak

ABSTRACT. We introduce Propositional Typicality Logic (PTL), a logic for reasoning about typicality. We do so by enriching classical propositional logic with a typicality operator of which the intuition is to capture the most typical (or normal) situations in which a given formula holds. The semantics is in terms of ranked models as studied in KLM-style preferential reasoning. This allows us to show that KLM-style rational consequence relations can be embedded in our logic. Moreover we show that we can define consequence relations on the language of PTL itself, thereby moving beyond the propositional setting. Building on the existing link between propositional rational consequence and belief revision, we show that the same correspondence holds in the case of rational consequence and belief revision defined on the language of PTL. Finally we also investigate different notions of entailment for PTL and propose two appropriate candidates.
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1 Introduction

In artificial intelligence, there has been a great deal of work done on how to introduce nonmonotonic reasoning capabilities in logic-based knowledge representation systems [Brachman and Levesque, 2004; Friedman and Halpern, 2001; Gabbay and Schlechta, 2009; Hansson, 1999; Harmelen *et al.*, 2008; Makinson, 2005]. In particular, the approach for *preferential reasoning* introduced by Shoham [1988] and developed by Kraus, Lehmann and Magidor [1990], often called the *KLM approach*, turned out to be one of the most successful. This has been the case due to at least three main reasons. Firstly, their framework is based on semantic constructions that are elegant and neat. Secondly, it provides the foundation for the determination of the important notion of entailment in this context [Lehmann and Magidor, 1992]. Finally, it also offers an alternative perspective on the problem of belief change [Gärdenfors and Makinson, 1994]. Moreover recent work has shown that the KLM approach also provides an appropriate springboard from which to investigate further facets of defeasible reasoning in more expressive logics [Britz *et al.*, 2008; Britz *et al.*, 2011a; Britz *et al.*, 2011b; Britz *et al.*, 2012; Britz *et al.*, 2013a; Britz *et al.*, 2013b; Britz and Varzinczak, 2012; Britz and Varzinczak, 2013; Casini and Straccia, 2010; Giordano *et al.*, 2009b; Lehmann and Magidor, 1990;

book title , 1–31.

Moodley *et al.*, 2012], such as modal logic [Chellas, 1980; Blackburn *et al.*, 2006] and description logics (DLs) [Baader *et al.*, 2007].

In their seminal papers [Kraus *et al.*, 1990; Lehmann and Magidor, 1992], Kraus and colleagues enrich classical propositional logic with a defeasible 'implication' $\mid\!\sim$ so that one can write down defeasible implication statements (also called conditional assertions) of the form $\alpha \mid\!\sim \beta$, where α and β are propositional formulae. In this setting, a sentence of the form $\alpha \mid\!\sim \beta$ is given the meaning that "if α is the case, then *usually* (but not necessarily always) β is the case", making it possible to formalize the well-known example that "birds usually fly" ($\mathsf{b} \mid\!\sim \mathsf{f}$), "penguins are birds" ($\mathsf{p} \rightarrow \mathsf{b}$), but "penguins usually do not fly" ($\mathsf{p} \mid\!\sim \neg\mathsf{f}$).

A curious aspect of the KLM approach (and of the corresponding belief revision constructions) is that it is crucially, albeit tacitly, based on a notion of *normality* [Boutilier, 1994] or *typicality* [Lehmann, 1998]. More formally, the semantics of a statement of the form $\alpha \mid\!\sim \beta$ says that "all most preferred (i.e., most normal) α-worlds are β-worlds" (leaving it open for α-worlds that are less preferred — or exceptional — not to satisfy β). In other words, the statement $\alpha \mid\!\sim \beta$ captures the intuition that "typical α-cases are β cases", which in our example gives us "typical birds fly" and "typical penguins do not fly". Given this, it seems quite natural to be able to state, for instance, that "penguins are non-typical birds", or that "penguins and ostriches are the only birds that typically do not fly".

It turns out that in the corresponding underlying language it is not possible to refer directly to such a notion of typicality and, importantly, use it in the scope of other logical constructs. According to Britz and Varzinczak [2012],

> This has to do partly with the syntactic restrictions imposed on $\mid\!\sim$, namely no nesting of conditionals, but, more fundamentally, it relates to where and how the notion of normality is used in such statements. ...[I]n a KLM defeasible statement $\alpha \mid\!\sim \beta$, the normality spotlight is somewhat put on α, as though normality was a property of the premise and not of the conclusion. Whether the situations in which β holds are normal or not plays no role in the reasoning that is carried out. In the original KLM framework, normality is linked to the premise as a whole, rather than its constituents. Technically this meant one could not refer directly to normality of a sentence in the scope of logical operators.

In this chapter, we fill this gap with the introduction of an explicit operator to talk about typicality. Intuitively, our new syntactic construction allows us to single out those most typical states of affairs in which a given formula holds. The result is a more expressive language allowing us, for instance, to make statements formalizing the aforementioned examples in a succinct way.

In the rest of this section we set up the notation and conventions that shall be followed in the upcoming sections. The remainder of the present chapter is then structured as follows: In Section 2 we provide the required background on the KLM approach to defeasible reasoning. We then define and investigate PTL, a propositional typicality logic extending propositional logic (Section 3). The semantics of PTL is in terms of ranked models as studied in the literature on preferential reasoning and summarized in Section 2. This allows us to embed propositional KLM-style consequence relations in our new language. In Section 4 we show that, although the addition of the typicality operator increases the expressivity of the logic, the nesting of the typicality does not add anything beyond the inclusion of a non-nested typicality operator. In Section 5 we investigate the link between AGM belief revision and PTL. We show that propositional AGM belief revision can be expressed in terms of typicality, and also that it can be lifted to a version of revision on PTL. We then move to an investigation of rational consequence relations in terms of PTL (Section 6). We show that propositional rational consequence can be expressed in PTL, that it can be extended to PTL in terms of PTL itself, and that the propositional connection between rational consequence and revision carries over to PTL. In Section 7 we raise the question of what an appropriate notion of entailment for PTL is. We propose and investigate different definitions of entailment and identify two appropriate candidates. After a discussion of and comparison with related work (Section 8), we conclude with a summary of the contributions and directions for further investigation.

1.1 Logical Preliminaries

We work in a propositional language over a finite set of propositional *variables* (alias *atoms*) $\mathcal{P}$. (In later sections we shall adopt a richer language.) We shall use $p, q, \ldots$ as meta-variables for the atomic propositions. Propositional formulae (and in later sections, formulae of the richer language) are denoted by $\alpha, \beta, \ldots$, and are recursively defined in the usual way: $\alpha ::= p \mid \neg\alpha \mid \alpha \wedge \alpha$. All the other Boolean truth-functional connectives ($\vee$, $\rightarrow$, $\leftrightarrow$, $\ldots$) are defined in terms of $\neg$ and $\wedge$ in the standard way. We use $\top$ as an abbreviation for $p \vee \neg p$, and $\bot$ for $p \wedge \neg p$, for some atom $p \in \mathcal{P}$. With $\mathcal{L}$ we denote the set of all propositional formulae.

We denote by $\mathscr{U}$ the set of all *valuations* $v : \mathcal{P} \longrightarrow \{0, 1\}$. Sometimes we shall represent the valuations of the logic under consideration as sequences of 0s and 1s, and with the obvious implicit ordering of atoms. Thus, for the logic generated from $\mathcal{P} = \{p, q\}$, the valuation in which p is true and q is false will be represented as 10.

Satisfaction of a formula $\alpha \in \mathcal{L}$ by $v \in \mathscr{U}$ is defined in the usual truth-functional way and is denoted by $v \Vdash \alpha$. With $Mod(\alpha)$ we denote the set of all valuations satisfying α. Logical consequence and logical equivalence are denoted by $\models$ and $\equiv$ respectively. Given sentences α and β, $\alpha \models \beta$ (α entails β) means $Mod(\alpha) \subseteq Mod(\beta)$. $\alpha \equiv \beta$ is an abbreviation of $\alpha \models \beta$ *and* $\beta \models \alpha$.

A *knowledge base* $\mathcal{K}$ is a finite set of formulae $\mathcal{K} \subseteq \mathcal{L}$. We extend the notions of $Mod(\cdot)$, entailment and logical equivalence to knowledge bases in the usual way: for a finite $\mathcal{K} \subseteq \mathcal{L}$, $Mod(\mathcal{K})$ is the set of all valuations satisfying every formula in $\mathcal{K}$, and $\mathcal{K} \models \alpha$ if and only if $Mod(\mathcal{K}) \subseteq Mod(\alpha)$. With $\models \alpha$ (α is a tautology) we understand $\emptyset \models \alpha$.

2 Defeasible Consequence Relations

In the present section, we provide a brief outline of propositional preferential and rational consequence relations as studied by Lehmann and colleagues in the early 90's with some minor modifications to their initial formulation. (For more details, the reader is referred to the original work of Kraus et al. [1990] and Lehmann and Magidor [1992].)

A *defeasible consequence relation* $\mid\!\sim$ is defined as a binary relation on formulae of our underlying propositional logic, i.e., $\mid\!\sim\, \subseteq \mathcal{L} \times \mathcal{L}$. We say that $\mid\!\sim$ is a *preferential* consequence relation [Kraus *et al.*, 1990] if it satisfies the following set of properties, alias postulates or Gentzen-style rules, as they are sometimes also referred to (below, $\models$ denotes validity in classical propositional logic):

(Ref) $\alpha \mid\!\sim \alpha$ (LLE) $\dfrac{\models \alpha \leftrightarrow \beta,\ \alpha \mid\!\sim \gamma}{\beta \mid\!\sim \gamma}$ (And) $\dfrac{\alpha \mid\!\sim \beta,\ \alpha \mid\!\sim \gamma}{\alpha \mid\!\sim \beta \wedge \gamma}$

(Or) $\dfrac{\alpha \mid\!\sim \gamma,\ \beta \mid\!\sim \gamma}{\alpha \vee \beta \mid\!\sim \gamma}$ (RW) $\dfrac{\alpha \mid\!\sim \beta,\ \models \beta \rightarrow \gamma}{\alpha \mid\!\sim \gamma}$ (CM) $\dfrac{\alpha \mid\!\sim \beta,\ \alpha \mid\!\sim \gamma}{\alpha \wedge \beta \mid\!\sim \gamma}$

The semantics of preferential consequence relations is in terms of *preferential models*; these are partially ordered structures with states labeled by propositional valuations. We make this terminology more precise below.

Let S be a set and $\prec\, \subseteq S \times S$ be a strict partial order on S, i.e., $\prec$ is *irreflexive* and *transitive*. Given $S' \subseteq S$, we say that $s \in S'$ is *minimal* in S' if there is no $s' \in S'$ such that $s' \prec s$. With $\min_{\prec} S'$ we denote the minimal elements of $S' \subseteq S$ with respect to $\prec$. We say that $S' \subseteq S$ is *smooth* [Kraus *et al.*, 1990] if for every $s \in S'$ either s is minimal in S' or there is $s' \in S'$ such that s' is minimal in S' and $s' \prec s$.

DEFINITION 1 *A preferential model is a tuple $\mathscr{P} = \langle S, \ell, \prec \rangle$ where S is a set of states; $\ell : S \longrightarrow \mathscr{U}$ is a labeling function; $\prec\, \subseteq S \times S$ is a strict partial order on S satisfying the smoothness condition.*[1]

Given a preferential model $\mathscr{P} = \langle S, \ell, \prec \rangle$ and $\alpha \in \mathcal{L}$, with $[\![\alpha]\!]^{\mathscr{P}}$ we denote the set of states satisfying α (α-*states* for short) in $\mathscr{P}$ according to the following definition:

DEFINITION 2 *Let $\mathscr{P} = \langle S, \ell, \prec \rangle$ be a preferential model and let $\alpha \in \mathcal{L}$. Then $[\![\alpha]\!]^{\mathscr{P}} := \{s \in S \mid \ell(s) \Vdash \alpha\}$.*

In the KLM approach, states lower down in the order are seen as being *more preferred* (or *more normal*) than those higher up.

As an example, let $\mathcal{P} = \{\mathsf{b}, \mathsf{f}, \mathsf{p}\}$, where b stands for the proposition "Tweety is a bird", f for "Tweety flies" and p for the proposition "Tweety is a penguin". Figure 1 below depicts the preferential model $\mathscr{P} = \langle S, \ell, \prec \rangle$ where $S = \{s_i \mid 1 \leq i \leq 6\}$, ℓ is such that $\ell(s_1) = 000$, $\ell(s_2) = 010$, $\ell(s_3) = 110$, $\ell(s_4) = 100$, $\ell(s_5) = 101$ and $\ell(s_6) = 111$, and $\prec$ is the transitive closure of $\{(s_1, s_4), (s_1, s_5), (s_2, s_4), (s_2, s_5), (s_3, s_4), (s_3, s_5), (s_4, s_6), (s_5, s_6)\}$.

[1] That is, for every $\alpha \in \mathcal{L}$, the set $[\![\alpha]\!]^{\mathscr{P}}$ (cf. Definition 2) is smooth.

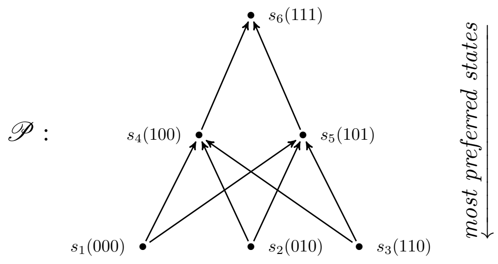

Figure 1. Example of a preferential model.

Given a preferential model $\mathscr{P} = \langle S, \ell, \prec \rangle$ and a sentence $\alpha \in \mathcal{L}$, we say that α is *satisfiable* in $\mathscr{P}$ if $[\![\alpha]\!]^{\mathscr{P}} \neq \emptyset$, otherwise α is *unsatisfiable* in $\mathscr{P}$. We say that α is *true* in $\mathscr{P}$ (denoted $\mathscr{P} \Vdash \alpha$) if $[\![\alpha]\!]^{\mathscr{P}} = S$.

Given $\mathscr{P} = \langle S, \ell, \prec \rangle$ and $\alpha, \beta \in \mathcal{L}$, the defeasible statement $\alpha \mid\!\sim \beta$ holds in $\mathscr{P}$ (noted $\alpha \mid\!\sim_{\mathscr{P}} \beta$) if and only if $\min_{\prec}[\![\alpha]\!]^{\mathscr{P}} \subseteq [\![\beta]\!]^{\mathscr{P}}$, i.e., every $\prec$-minimal α-state is a β-state. As an example, in the model $\mathscr{P}$ of Figure 1, we have $\mathsf{b} \mid\!\sim_{\mathscr{P}} \mathsf{f}$ (since $\min_{\prec}[\![\mathsf{b}]\!]^{\mathscr{P}} = \{s_3\} \subseteq [\![\mathsf{f}]\!]^{\mathscr{P}} = \{s_2, s_3, s_6\}$), and $\mathsf{p} \mid\!\sim_{\mathscr{P}} \neg\mathsf{f}$ (since $\min_{\prec}[\![\mathsf{p}]\!]^{\mathscr{P}} = \{s_5\} \subseteq [\![\neg\mathsf{f}]\!]^{\mathscr{P}} = \{s_1, s_4, s_5\}$).

The representation theorem for preferential consequence relations then states:

THEOREM 3 (Kraus et al. [1990]) *A defeasible consequence relation is a preferential consequence relation if and only if it is defined by some preferential model, i.e.,* $\mid\!\sim$ *is preferential if and only if there exists* $\mathscr{P}$ *such that* $\mid\!\sim_{\mathscr{P}} := \{(\alpha, \beta) \mid \alpha \mid\!\sim_{\mathscr{P}} \beta\}$ *is such that* $\mid\!\sim \, = \, \mid\!\sim_{\mathscr{P}}$.

If, in addition to the preferential properties, the defeasible consequence relation $\mid\!\sim$ also satisfies the following Rational Monotonicity property [Lehmann and Magidor, 1992], it is said to be a *rational* consequence relation:

$$(\text{RM}) \ \frac{\alpha \mid\!\sim \beta, \quad \alpha \not\mid\!\sim \neg\gamma}{\alpha \wedge \gamma \mid\!\sim \beta}$$

The semantics of rational consequence relations is in terms of *ranked* models, i.e., preferential models in which the preference order is *modular*:

DEFINITION 4 *Given a set* S, $\prec \subseteq S \times S$ *is modular if and only if there is a ranking function* $rk : S \longrightarrow \mathbb{N}$ *such that for every* $s, s' \in S$, $s \prec s'$ *if and only if* $rk(s) < rk(s')$.

DEFINITION 5 *A ranked model* $\mathscr{R} = \langle S, \ell, \prec \rangle$ *is a preferential model such that* $\prec$ *is modular.*

The preferential model in Figure 1 is also an example of a ranked model.

The representation theorem for rational consequence relations then states:

THEOREM 6 (Lehmann & Magidor [1992]) *A defeasible consequence relation is a rational consequence relation if and only if it is defined by some ranked model, i.e.,* $\mid\!\sim$ *is rational if and only if there exists* $\mathscr{R}$ *such that* $\mid\!\sim_{\mathscr{R}} := \{(\alpha, \beta) \mid \alpha \mid\!\sim_{\mathscr{R}} \beta\}$ *is such that* $\mid\!\sim\, = \,\mid\!\sim_{\mathscr{R}}$.

There seems to be an agreement in the nonmonotonic reasoning community that rational consequence constitutes the 'right' type of entailment for nonmonotonic logics; one of the reasons stemming from its confluence with the AGM paradigm for belief revision [Alchourrón *et al.*, 1985; Hansson, 1999] (see also Section 5). In the remainder of this chapter we shall therefore assume that $\mid\!\sim$ is at least rational.[2]

From a technical point of view, the main advantage of assuming rationality is that we can do away with states and work with a much simpler semantics in which the preferential ordering is placed directly on valuations [Gärdenfors and Makinson, 1994]. Therefore, from now on we shall adopt the following definition of ranked models:

DEFINITION 7 *A ranked model* $\mathscr{R}$ *is a pair* $\langle \mathcal{V}, \prec \rangle$, *where* $\mathcal{V} \subseteq \mathscr{U}$ *and* $\prec \subseteq \mathcal{V} \times \mathcal{V}$ *is a modular order over* $\mathcal{V}$.

The definition above is of course not Lehmann and Magidor's [1992] original definition of ranked models, but as alluded to above, a characterization of rational consequence *à la* Theorem 6 can be given in terms of ranked models as we present them here [Gärdenfors and Makinson, 1994].

DEFINITION 8 *Let* $\alpha \in \mathcal{L}$ *and let* $\mathscr{R} = \langle \mathcal{V}, \prec \rangle$ *be a ranked model. With* $[\![\alpha]\!]$ *we denote the set of valuations satisfying* α *in* $\mathscr{R}$, *defined as follows:*

$$[\![p]\!] := \{v \in \mathcal{V} \mid v(p) = 1\}, \quad [\![\neg\alpha]\!] := \mathcal{V} \setminus [\![\alpha]\!], \quad [\![\alpha \wedge \beta]\!] := [\![\alpha]\!] \cap [\![\beta]\!]$$

Given a (simplified) ranked model $\mathscr{R}$, as in the case with states, the intuition is that valuations lower down in the ordering are more preferred than those higher up. Hence, a pair (α, β) is in the consequence relation defined by $\mathscr{R}$ (denoted as $\alpha \mid\!\sim_{\mathscr{R}} \beta$) if and only if $\min_{\prec}[\![\alpha]\!] \subseteq [\![\beta]\!]$, i.e., the most preferred (with respect to $\prec$) α-valuations are also β-valuations.

3 A Logic to Talk about Typicality

As alluded to in the Introduction, there is a need for a formalism in which we can make statements such as "the typical bird-situations", "ostriches are non-typical birds", and "penguins and ostriches are the only typical non-flying birds". In some cases, it may be possible to express these by means of several $\mid\!\sim$-statements, in a sort of '$\mid\!\sim$-normal form'. However, in order for us to do so in a succinct way, we should be able to shift the focus of typicality from the premise of a KLM-style statement and drop the interdiction to nest $\mid\!\sim$'s.

Boutilier's [1994] conditional $\Rightarrow$ as well as Britz et al.'s [2009] internalized $\prec$ as a modality are good candidates for the type of extension that we have in mind

[2]Even in the context of rational consequence relations, in what follows we shall use preferential semantics and preferential reasoning when referring to the KLM approach.

here. Nevertheless, these approaches are too expressive for our purposes in that there the preference relation $\prec$ becomes explicit to the user (cf. Section 8). Here we argue for a way to express typicality in which the complexity of the underlying semantics, here expressed as the preference relation $\prec$, is somehow hidden from the users, who, from a knowledge representation perspective, want a formalism which is precise but at the same time concise. (We shall come back to this point at the end of this section.)

The remainder of the present section is devoted to the introduction of the main focus of this chapter, namely a propositional typicality logic, called PTL, which extends classical propositional logic with a *typicality operator* $\bullet$, the semantics of which *implicitly* refers to the preference ordering.

The language of PTL, denoted by $\mathcal{L}^\bullet$, is recursively defined as follows:

$$\alpha \ ::= p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \bullet\alpha$$

where, as before, p denotes an atom and all the other connectives are defined in terms of $\neg$ and $\wedge$, and $\top$ and $\bot$ are seen as abbreviations. Assuming $\mathcal{P} = \{\mathsf{b}, \mathsf{f}, \mathsf{p}, \mathsf{o}\}$, where b, f and p are as before and o stands for "is an ostrich", the following are examples of $\mathcal{L}^\bullet$-sentences: $\bullet\mathsf{b}$, $\mathsf{o} \rightarrow \neg\bullet\mathsf{b}$, $\mathsf{p} \vee \mathsf{o} \leftrightarrow \mathsf{b} \wedge \bullet\neg\mathsf{f}$.

Intuitively, a sentence of the form $\bullet\alpha$ is understood to refer to the typical situations in which α holds. (Note that α can itself be a $\bullet$-sentence — more on that in Section 4.) The semantics of our enriched language is in terms of (simplified) ranked models (cf. Definition 7) and we extend the notion of satisfaction from Definition 8 as follows:

DEFINITION 9 *Let* $\alpha \in \mathcal{L}^\bullet$ *and let* $\mathscr{R} = \langle \mathcal{V}, \prec \rangle$. *Then* $[\![\bullet\alpha]\!] := \min_\prec [\![\alpha]\!]$.

Given $\alpha \in \mathcal{L}^\bullet$ and $\mathscr{R}$ a ranked model, we say that α is *satisfiable* in $\mathscr{R}$ if $[\![\alpha]\!] \neq \emptyset$, otherwise α is *unsatisfiable* in $\mathscr{R}$. We say that α is *true* in $\mathscr{R}$ (denoted as $\mathscr{R} \Vdash \alpha$) if $[\![\alpha]\!] = \mathcal{V}$. For $\mathcal{K} \subseteq \mathcal{L}^\bullet$, $\mathscr{R} \Vdash \mathcal{K}$ if $\mathscr{R} \Vdash \alpha$ for every $\alpha \in \mathcal{K}$. We say that α is *valid*, denoted as $\models \alpha$, if $\mathscr{R} \Vdash \alpha$ for every ranked model $\mathscr{R}$.[3]

As an example, let $\mathcal{P} = \{\mathsf{b}, \mathsf{f}, \mathsf{p}\}$ and consider the (simplified) ranked model $\mathscr{R}$ depicted in Figure 2. Then we have $[\![\bullet\mathsf{b}]\!] = \{110\}$, $[\![\bullet\mathsf{p}]\!] = \{101\}$ and $[\![\bullet(\mathsf{b} \wedge \neg\mathsf{f})]\!] = \{100, 101\}$.

It is worth noting that for every ranked model $\mathscr{R}$ and every $\alpha \in \mathcal{L}^\bullet$, there is a $\beta \in \mathcal{L}$ (i.e., a propositional formula) such that $\mathscr{R} \Vdash \alpha \leftrightarrow \beta$. That is to say, given $\mathscr{R}$, every α can be expressed as a propositional formula (β) in $\mathscr{R}$. Of course, this does not mean that propositional logic is as expressive as PTL, since the formula β used to express α in the ranked model $\mathscr{R}$ depends on the specific $\mathscr{R}$. Rather, the relationship between PTL and propositional logic is similar to the relationship between modal logic and propositional logic in the sense that both modal logic and PTL add to propositional logic an operator that is not truth-functional. (In Section 8 we discuss in more detail the relationship between PTL and modal approaches to preferential reasoning.)

[3]The observant reader would have noticed that this sounds like modal logic [Chellas, 1980]. We shall defer a discussion on our typicality operator as a modality until the appropriate point (Section 8).

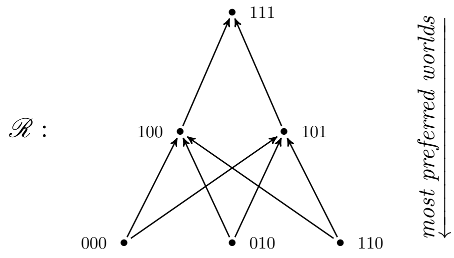

Figure 2. An $\mathcal{L}^{\bullet}$-model for $\mathcal{P} = \{\mathsf{b}, \mathsf{f}, \mathsf{p}\}$.

Next is a result showing that if a ranked model $\mathscr{R}$ is such that $\mathscr{R} \Vdash \bullet\alpha$ for some $\alpha \in \mathcal{L}^{\bullet}$, then $\mathscr{R}$ consists of only α-worlds in which all worlds are incomparable (alias equally preferred) according to the preference relation $\prec$. In other words, typicality allows a syntactic way of expressing the preference relation is empty.

PROPOSITION 10 *Let* $\mathscr{R} = \langle \mathcal{V}, \prec \rangle$. *Then*

1. $\prec = \emptyset$ *if and only if there exists* $\alpha \in \mathcal{L}^{\bullet}$ *such that* $\mathscr{R} \Vdash \bullet\alpha$*;*

2. *For every* $\alpha \in \mathcal{L}^{\bullet}$, $\mathscr{R} \Vdash \bullet\alpha$ *if and only if for every* $\beta \in \mathcal{L}$ *such that* $\mathscr{R} \Vdash \alpha \to \beta$, $\mathscr{R} \Vdash \bullet\beta$.

Proof.
Proving Part 1: Suppose that $\prec = \emptyset$. Then it is not hard to verify that $\mathscr{R} \Vdash \bullet\top$. Conversely, suppose that $\mathscr{R} \Vdash \bullet\alpha$ for some $\alpha \in \mathcal{L}^{\bullet}$. That is, $\min_{\prec}[\![\alpha]\!] = \mathcal{V}$. From this it follows that $\prec = \emptyset$ since otherwise there would be a valuation $v \in \mathcal{V}$ such that $v \notin \min_{\prec}[\![\alpha]\!]$.

Proving Part 2: Pick an $\alpha \in \mathcal{L}^{\bullet}$ and suppose $\mathscr{R} \Vdash \bullet\alpha$. Therefore $\mathscr{R} \Vdash \alpha$ and since $\mathscr{R} \Vdash \alpha \to \beta$, it follows that $\mathscr{R} \Vdash \beta$. Now, from $\mathscr{R} \Vdash \bullet\alpha$ it also follows that $\prec = \emptyset$, and from $\mathscr{R} \Vdash \beta$ we then have that $\mathscr{R} \Vdash \bullet\beta$. Conversely, suppose that for every $\beta \in \mathcal{L}$ such that $\mathscr{R} \Vdash \alpha \to \beta$, $\mathscr{R} \Vdash \bullet\beta$. Let β be α itself, from which it then follows that $\mathscr{R} \Vdash \bullet\alpha$. ■

One of the consequences of this result is that if $\bullet\alpha$ is true in a ranked model, then so is α (but the converse, of course, does not hold).

Another useful property of the typicality operator $\bullet$ is that it allows us to express (propositional) rational consequence, as defined in Section 2.

PROPOSITION 11 *Let* $\mathscr{R} = \langle \mathcal{V}, \prec \rangle$. *Then for every* $\alpha, \beta \in \mathcal{L}$ *(i.e.,* α *and* β *are propositional formulae),* $\alpha \mid\!\sim_{\mathscr{R}} \beta$ *if and only if* $\mathscr{R} \Vdash \bullet\alpha \to \beta$.

Proof. Let $\mathscr{R} = \langle \mathcal{V}, \prec \rangle$ be a ranked model. $\alpha \mid\!\sim_{\mathscr{R}} \beta$ if and only if $\min_{\prec}[\![\alpha]\!] \subseteq [\![\beta]\!]$ if and only if $\min_{\prec}[\![\alpha]\!] \cap [\![\neg\beta]\!] = \emptyset$ if and only if $[\![\bullet\alpha \wedge \neg\beta]\!] = \emptyset$ if and only if $[\![\bullet\alpha \to \beta]\!] = \mathcal{V}$ if and only if $\mathscr{R} \Vdash \bullet\alpha \to \beta$. ■

Proposition 11 shows that the introduction of a typicality operator into the object language allows us to express KLM-style rational consequence as defined in Section 2.[4] This forms part of our argument to show that our semantics for typicality is the correct one, but it does not provide a justification for introducing all the additional expressivity obtained from typicality. The next two results provide such a justification.

PROPOSITION 12 *There is an $\mathcal{L}^{\bullet}$-sentence that cannot be expressed as a single KLM-style $\mid\!\sim$-statement. That is, there is $\alpha \in \mathcal{L}^{\bullet}$ such that there exists $\mathscr{R}$ such that $\mathscr{R} \Vdash \alpha$ and for every $\beta, \gamma \in \mathcal{L}$, $\beta \not\mid\!\sim_{\mathscr{R}} \gamma$.*

The sentence $\alpha = q \wedge (\bullet q \to \neg p)$ is such that it cannot be expressed as a *single* $\mid\!\sim$-statement. The proof is tedious and we shall omit it here. However, note that α can be expressed as the *set* of defeasible statements $\{\neg q \mid\!\sim \bot, q \mid\!\sim \neg p\}$.

This raises the question whether the $\mathcal{L}^{\mid\sim}$-language, i.e., the set of all $\mid\!\sim$-statements built up from a propositional language $\mathcal{L}$, is as expressive as $\mathcal{L}^{\bullet}$. This is so if and only if for every $\alpha \in \mathcal{L}^{\bullet}$ there is a subset X of $\mathcal{L}^{\mid\sim}$ such that, for every ranked model $\mathscr{R}$, $\mathscr{R} \Vdash \alpha$ if and only if $\mathscr{R} \Vdash X$. The answer is 'no', as witnessed by the following result.

PROPOSITION 13 *There are $\mathcal{L}^{\bullet}$-sentences that cannot be expressed as a set of KLM-style $\mid\!\sim$-statements.*

Proof. Assume $\mathcal{P} = \{p, q\}$ and let α be the sentence $\bullet p$. This sentence is true in exactly four ranked models (all of which have empty orderings): one with $\mathcal{V} = \{11, 10\}$, one with $\mathcal{V} = \{11\}$, one with $\mathcal{V} = \{10\}$ and one with $\mathcal{V} = \emptyset$. Assume that there is a set X of $\mid\!\sim$-statements that are all true in exactly these ranked models. Then for each ranked model built up from $\mathcal{P}$ other than these four, there exists at least one $\mid\!\sim$-statement contained in X that is not true in it. In particular there exists such a $\mid\!\sim$-statement, call it $\beta \mid\!\sim \gamma$, for the ranked model in which $\mathcal{V} = \{11, 10\}$ and 11 is preferred to 10. Since $\beta \mid\!\sim \gamma$ is not true in this model, there is a world in $\min_{\prec}[\![\beta]\!]$ which is not in $[\![\gamma]\!]$. If 11 is such a world, then $\beta \mid\!\sim \gamma$ cannot be true in the ranked model $\langle\{11\}, \emptyset\rangle$ mentioned above. (Remember that β and γ are both propositional and therefore their truth values are entirely determined by the propositional valuations.) Alternatively, if 10 is such a world, then $\beta \mid\!\sim \gamma$ cannot be true in $\langle\{10\}, \emptyset\rangle$ either. Hence there cannot be such a world falsifying the statement $\beta \mid\!\sim \gamma$, and therefore X does not rule out the ranked models with a non-empty ordering, from which we derive a contradiction. Hence there cannot be such a set X. ■

A corollary of this result is that PTL does indeed add to the expressivity of the KLM approach. In the next section we assess how much expressivity is actually added by our typicality operator.

[4] Observe that Proposition 11 shows that rational consequence for *propositional* logic can be expressed in PTL. In Section 6 we shall see that it is also possible to express, in PTL itself, the extended notion of rational consequence for the more expressive language of $\mathcal{L}^{\bullet}$.

4 Unnesting the Birds

In the previous section we have argued for the need to include typicality explicitly in the object language. The observant reader would have noticed that the language of PTL allows for any arbitrary (finite) nesting of the typicality operator. An important point to consider is whether this much expressivity is needed, and whether it is not perhaps sufficient to restrict the language to non-nested applications of typicality.

In this section we show that once typicality is added to the language, nesting does not increase the expressivity any further, provided that we are allowed to add new propositional atoms. We shall thus be working with languages in which the set of propositional atoms $\mathcal{P}$ may vary, and more specifically, with languages with respect to a given knowledge base. So, given a knowledge base $\mathcal{K}$, we denote by $\mathcal{P}^{\mathcal{K}}$ the set of propositional atoms occurring in $\mathcal{K}$. Furthermore, by a *ranked model on* $\mathcal{P}^{\mathcal{K}}$ we mean a ranked model built up using only the propositional atoms occurring in $\mathcal{P}^{\mathcal{K}}$.

Now, given any finite $\mathcal{K} \subseteq \mathcal{L}^{\bullet}$ we: (*i*) Show how to transform every sentence $\beta \in \mathcal{L}^{\bullet}$ into a new sentence $\widehat{\beta}$ containing no nested instances of the $\bullet$ operator (and therefore also how to transform $\mathcal{K}$ into a knowledge base $\widehat{\mathcal{K}}$, containing no nested instances of the $\bullet$ operator); (*ii*) Show how to construct an auxiliary set of formulae $\widehat{E}$, containing no nested instances of the $\bullet$ operator, regulating the behavior of the newly introduced propositional atoms, and (*iii*) Show how to transform every ranked model $\mathscr{R}$ on $\mathcal{P}^{\mathcal{K}}$ into its 'appropriate representative' $\widehat{\mathscr{R}}$ on the set of atoms $\mathcal{P}^{\widehat{\mathcal{K}}}$ such that, for every $\beta \in \mathcal{L}^{\bullet}$, β is true in $\mathscr{R}$ if and only if $\widehat{\beta}$ is true in $\widehat{\mathscr{R}}$. Using these constructions we show that $\widehat{\mathcal{K}} \cup \widehat{E}$ is the non-nested version of the original knowledge base $\mathcal{K}$ in the sense that the ranked models in which $\widehat{\mathcal{K}} \cup \widehat{E}$ are true are precisely the 'appropriate representatives' of the ranked models in which $\mathcal{K}$ is true.

To be more precise, let $\mathcal{K} \subseteq \mathcal{L}^{\bullet}$ be a knowledge base, let $S^{\mathcal{K}}$ denote all the subformulae of $\mathcal{K}$, and let $B^{\mathcal{K}} := \{\bullet\alpha \in S^{\mathcal{K}} \mid \alpha \in \mathcal{L}\}$. So $B^{\mathcal{K}}$ contains all occurrences of subformulae in $\mathcal{K}$ containing a single $\bullet$ operator. Informally, the idea is to substitute (all occurrences of) every element $\bullet\alpha$ of $B^{\mathcal{K}}$ with a newly introduced atom $p^{\bullet\alpha}$, and to require that $p^{\bullet\alpha}$ be equivalent to $\bullet\alpha$. In doing so we reduce the level of nesting in $\mathcal{K}$ by a factor of 1. Now, let $E^{\mathcal{K}} := \{p^{\bullet\alpha} \leftrightarrow \bullet\alpha \mid \bullet\alpha \in B^{\mathcal{K}}\}$, and for every $\beta \in \mathcal{L}^{\bullet}$, let $\beta^{\mathcal{K}}$ be obtained from β by the simultaneous substitution in β of (every occurrence of) every $\bullet\alpha \in B^{\mathcal{K}}$ by $p^{\bullet\alpha}$ (observe that $\beta^{\mathcal{K}} = \beta$ if β is a propositional formula). We refer to $\beta^{\mathcal{K}}$ as the $\mathcal{K}$-*transform of* β. Also, let $\mathcal{K}^{\bullet} := \{\beta^{\mathcal{K}} \mid \beta \in \mathcal{K}\}$. The idea is that $\mathcal{K}^{\bullet} \cup E^{\mathcal{K}}$ is a version of $\mathcal{K}$ with one fewer level of nesting.

EXAMPLE 14 *Let* $\mathcal{K} = \{\bullet(p \wedge q) \rightarrow r, \bullet(\bullet p \vee r), \bullet(p \wedge \bullet(q \vee \bullet r))\}$. *Then we have* $S^{\mathcal{K}} = \mathcal{K} \cup \{\bullet(p \wedge q), p, q, r, \bullet p, \bullet(q \vee \bullet r), \bullet r\}$. *Then* $B^{\mathcal{K}} = \{\bullet(p \wedge q), \bullet p, \bullet r\}$ *and* $E^{\mathcal{K}} = \{p^{\bullet(p\wedge q)} \leftrightarrow \bullet(p \wedge q), p^{\bullet p} \leftrightarrow \bullet p, p^{\bullet r} \leftrightarrow \bullet r\}$. *Now* $(\bullet(p \wedge q) \rightarrow r)^{\mathcal{K}} = p^{\bullet(p\wedge q)} \rightarrow r$, $(\bullet(\bullet p \vee r))^{\mathcal{K}} = \bullet(p^{\bullet p} \vee r)$, $(\bullet(p \wedge \bullet(q \vee \bullet r)))^{\mathcal{K}} = \bullet(p \wedge \bullet(q \vee p^{\bullet r}))$. *Hence* $\mathcal{K}^{\bullet} = \{p^{\bullet(p\wedge q)} \rightarrow r, \bullet(p^{\bullet p} \vee r), \bullet(p \wedge \bullet q \vee p^{\bullet r})\}$. *Observe that* $\mathcal{K}$ *has a nesting level of 3, while* $\mathcal{K}^{\bullet}$ *has a nesting level of 2.*

Let $\mathscr{R} = \langle \mathcal{V}, \prec \rangle$ be a ranked model on $\mathcal{P}^{\mathcal{K}}$. We define $\mathscr{R}^{\bullet} = \langle \mathcal{V}^{\bullet}, \prec^{\bullet} \rangle$ on $\mathcal{P}^{\mathcal{K}^{\bullet}}$

as follows: for all $v \in \mathcal{V}$, let $v^\bullet$ be a valuation on $\mathcal{P}^{\mathcal{K}^\bullet}$ such that (i) $v^\bullet(p) = v(p)$ for every $p \in \mathcal{P}^{\mathcal{K}}$, and (ii) for every $p^{\bullet\alpha} \in \mathcal{P}^{\mathcal{K}^\bullet} \setminus \mathcal{P}^{\mathcal{K}}$, $v^\bullet(p^{\bullet\alpha}) = 1$ if and only if $v \in [\![\bullet\alpha]\!]$ in the ranked model $\mathscr{R}$. And for all $v^\bullet, v^{\bullet\prime} \in \mathcal{V}^\bullet$, $v^\bullet \prec^\bullet v^{\bullet\prime}$ if and only if $v \prec v'$. So $\mathscr{R}^\bullet$ is an extended version of $\mathscr{R}$ with every valuation v in $\mathscr{R}$ replaced with an extended valuation $v^\bullet$ in which the truth values of the atoms occurring in v remain unchanged, and the truth values of the new atoms are constrained by the requirement that every $p^{\bullet\alpha}$ be equivalent to $\bullet\alpha$ (for $\bullet\alpha \in B^{\mathcal{K}}$). We refer to $\mathscr{R}^\bullet$ as the *$\mathcal{K}$-extended version of $\mathscr{R}$*. From this we obtain the following result.

PROPOSITION 15 *Let $\mathcal{K}$ be an $\mathcal{L}^\bullet$-knowledge base. Then*

1. *For every ranked model $\mathscr{R}$ on $\mathcal{P}^{\mathcal{K}}$, $\mathscr{R}^\bullet \Vdash E^{\mathcal{K}}$;*
2. *A ranked model $\mathscr{R}'$ on $\mathcal{P}^{\mathcal{K}^\bullet}$ is such that $\mathscr{R}' \Vdash E^{\mathcal{K}}$ if and only if there is a ranked model $\mathscr{R}$ on $\mathcal{P}^{\mathcal{K}}$ such that $\mathscr{R}^\bullet = \mathscr{R}'$;*
3. *For all $\beta \in \mathcal{L}^\bullet$, $\mathscr{R} \Vdash \beta$ if and only if $\mathscr{R}^\bullet \Vdash \beta^{\mathcal{K}}$.*

Proof. Let $\mathcal{K} \subseteq \mathcal{L}^\bullet$ be finite.

Proving 1: Let $\mathscr{R}$ be a ranked model on $\mathcal{P}^{\mathcal{K}}$ and consider $\mathscr{R}^\bullet = \langle \mathcal{V}^\bullet, \prec^\bullet \rangle$. Pick any $p^{\bullet\alpha} \leftrightarrow \bullet\alpha \in E^{\mathcal{K}}$ and pick any $v \in \mathcal{V}^\bullet$. By construction, $v(p^{\bullet\alpha}) = 1$ if and only if $v \in [\![\bullet\alpha]\!]$, and so $v \in [\![p^{\bullet\alpha} \leftrightarrow \bullet\alpha]\!]$. From this it follows that $\mathscr{R} \Vdash E^{\mathcal{K}}$.

Proving 2: Pick a ranked model $\mathscr{R}'$ on $\mathcal{P}^{\mathcal{K}^\bullet}$ and suppose that $\mathscr{R}' \Vdash E^{\mathcal{K}}$ is the case. Let $\mathscr{R} = \langle \mathcal{V}, \prec \rangle$ be the ranked model on $\mathcal{P}^{\mathcal{K}}$ obtained from $\mathscr{R}'$ by restricting $\mathcal{V}$ (and therefore $\prec$ as well) to $\mathcal{P}^{\mathcal{K}}$. It follows immediately that the ranked model $\mathscr{R}^\bullet$ on $\mathcal{P}^{\mathcal{K}^\bullet}$ obtained from $\mathscr{R}$ is therefore equal to $\mathscr{R}'$. Conversely, suppose there is a ranked model $\mathscr{R}$ on $\mathcal{P}^{\mathcal{K}}$ such that $\mathscr{R}^\bullet = \mathscr{R}'$. By construction, $v^\bullet(p^{\bullet\alpha}) = 1$ if and only if $v^\bullet \in [\![\bullet\alpha]\!]$ for every $p^{\bullet\alpha} \in \mathcal{P}^{\mathcal{K}^\bullet} \setminus \mathcal{P}^{\mathcal{K}}$, from which it follows that $\mathscr{R}^\bullet \Vdash p^{\bullet\alpha} \leftrightarrow \bullet\alpha$ for every $p^{\bullet\alpha} \leftrightarrow \bullet\alpha \in E^{\mathcal{K}}$. So $\mathscr{R}^\bullet$, and therefore $\mathscr{R}'$, is a model of $E^{\mathcal{K}}$.

Proving 3: Pick any ranked model $\mathscr{R} = \langle \mathcal{V}, \prec \rangle$ on $\mathcal{P}^{\mathcal{K}}$. Suppose that $\mathscr{R} \Vdash \mathcal{K}$ and pick any $\beta \in \mathcal{L}$. Pick a $v \in \mathcal{V}$ and suppose that $v \in [\![\beta]\!]$. Now consider $\mathscr{R}^\bullet = \langle \mathcal{V}^\bullet, \prec^\bullet \rangle$ and, in particular, $v^\bullet \in \mathcal{V}^\bullet$. From the construction of $v^\bullet$ and $\beta^{\mathcal{K}}$ it follows immediately that $v^\bullet \in [\![\beta^{\mathcal{K}}]\!]$ in $\mathscr{R}^\bullet$. Conversely, suppose that $\mathscr{R}^\bullet \Vdash \mathcal{K}^\bullet$ and pick any $\beta \in \mathcal{L}^\bullet$. Pick a $v^\bullet \in \mathcal{V}^\bullet$ and suppose that $v^\bullet \in [\![\beta^{\mathcal{K}}]\!]$. Now consider $\mathscr{R} = \langle \mathcal{V}, \prec \rangle$ and, in particular, $v \in \mathcal{V}$. From the construction of $v^\bullet$ and $\bullet\beta$ it follows immediately that $v \in [\![\beta^{\mathcal{K}}]\!]$. ■

Proposition 15 above shows that the $\mathcal{K}$-extended version of a ranked model $\mathscr{R}$ is the only 'appropriate representative' of $\mathscr{R}$ in the class of ranked models based on the extended language of $\mathcal{K}^\bullet$. In addition, the $\mathcal{K}$-extended versions of the ranked models based on the language of $\mathcal{K}$ are the only ones satisfying $E^{\mathcal{K}}$.

From Proposition 15 it also follows that all sentences β of $\mathcal{L}^\bullet$ and their $\mathcal{K}$-transforms behave exactly the same with respect to, respectively, any ranked model $\mathscr{R}$ and its $\mathcal{K}$-extended version $\mathscr{R}^\bullet$. This applies, in particular, to the elements of $\mathcal{K}$, as the next corollary shows.

COROLLARY 16 *Let $\mathcal{K}$ be an $\mathcal{L}^{\bullet}$-knowledge base and let $\mathscr{R}$ be a ranked model on $\mathcal{P}^{\mathcal{K}}$. Then $\mathscr{R} \Vdash \mathcal{K}$ if and only if $\mathscr{R}^{\bullet} \Vdash \mathcal{K}^{\bullet}$.*

Proof. Follows from Part 3 of Proposition 15. ■

These results show that the move from a knowledge base $\mathcal{K}$ to $\mathcal{K}^{\bullet}$ ensures that we can reduce the level of nesting of $\bullet$ operators by a factor of 1. To arrive at a set $\widehat{\mathcal{K}}$ not containing any nested occurrences of $\bullet$ we just need to iterate the transform process a sufficient number of times. So, we define $\widehat{\mathcal{K}}$ as follows: Let $\mathcal{K}_0 := \mathcal{K}$, and for every $i > 0$, let $B_i := B^{\mathcal{K}_{i-1}}$, $\mathcal{K}_i := \mathcal{K}_{i-1}^{\bullet}$, and let $n := \min_{<}\{i \mid B_{i+1} = \emptyset\}$. We then let $\widehat{\mathcal{K}} := \mathcal{K}_n$. So for every $i = 1, \ldots, n$, $\mathcal{K}_i$ has one fewer level of nesting of $\bullet$ than $\mathcal{K}_{i-1}$ until we get to $\mathcal{K}_n = \widehat{\mathcal{K}}$, which has no nested occurrences of $\bullet$. Similarly, for every $\beta \in \mathcal{L}^{\bullet}$, we define $\widehat{\beta}$ as follows: Let $\beta_0 := \beta$, for every $i = 1, \ldots, n$, let $\beta_i := \beta^{\mathcal{K}_{i-1}}$, and let $\widehat{\beta} := \beta_n$. We refer to $\widehat{\beta}$ as the *full $\mathcal{K}$-transform of β*. In a similar vein, we let $\widehat{E} := \bigcup_{i=0}^{i=n-1} E^{\mathcal{K}_i}$.

EXAMPLE 17 *Continuing Example 14, let $\mathcal{K}_0 = \mathcal{K}$. Then $B_1 = B^{\mathcal{K}_0} = B^{\mathcal{K}}$, and $\mathcal{K}_1 = \mathcal{K}^{\bullet}$ with $E_0 = E^{\mathcal{K}}$. Then*

$$S^{\mathcal{K}_1} = S^{\mathcal{K}^{\bullet}} = \mathcal{K}^{\bullet} \cup \{p^{\bullet(p\wedge q)}, r, p^{\bullet p}, p, \bullet(q \vee p^{\bullet r}), q, p^{\bullet r}\}$$

and then $B_2 = B^{\mathcal{K}_1} = \{\bullet(q \vee p^{\bullet r})\}$, and $E_1 = \{p^{\bullet(q\vee p^{\bullet r})} \leftrightarrow \bullet(q \vee p^{\bullet r})\}$. Now we have $\mathcal{K}_2 = \mathcal{K}_1^{\bullet} = \mathcal{K}^{\bullet\bullet} = \{p^{\bullet(p\wedge q)} \to r, \bullet(p^{\bullet p} \vee r), \bullet(p \wedge p^{\bullet(q\vee p^{\bullet r})})\}$. Then, in the second iteration, we get

$$S^{\mathcal{K}_2} = \mathcal{K}_2 \cup \{p^{\bullet(p\wedge q)}, r, p^{\bullet r}, p, p^{\bullet(q\vee p^{\bullet r})}\}$$

and then $B_3 = B^{\mathcal{K}_2} = \{\bullet(p^{\bullet p} \vee r), \bullet(p \wedge p^{\bullet(q\vee p^{\bullet r})})\}$ with $E_2 = \{p^{\bullet(p^{\bullet p}\vee r)} \leftrightarrow \bullet(p^{\bullet p}\vee r), p^{\bullet(p\wedge p^{\bullet(q\vee p^{\bullet r})})} \leftrightarrow \bullet(p\wedge p^{\bullet(q\vee p^{\bullet r})})\}$. Then we get $\mathcal{K}_3 = \mathcal{K}_2^{\bullet} = \{p^{\bullet(p\wedge q)} \to r, p^{\bullet(p^{\bullet p}\vee r)}, p^{\bullet(p\wedge p^{\bullet(q\vee p^{\bullet r})})}\}$. In the next iteration, we have

$$S^{\mathcal{K}_3} = \mathcal{K}_3 \cup \{p^{\bullet(p\wedge q)}, r, p^{\bullet(p^{\bullet p}\vee r)}, p^{\bullet(p\wedge p^{(\bullet q\vee p^{\bullet r})})}\}$$

and therefore $B_4 = \emptyset$. Hence $n = 3$, and then

$$\widehat{\mathcal{K}} = \left\{ \; p^{\bullet(p\wedge q)} \to r, \; p^{\bullet(p^{\bullet p}\vee r)}, \; p^{\bullet(p\wedge p^{\bullet(q\vee p^{\bullet r})})} \; \right\}$$

$$\widehat{E} = \left\{ \begin{array}{c} p^{\bullet(p\wedge q)} \leftrightarrow \bullet(p \wedge q), \; p^{\bullet p} \leftrightarrow \bullet p, \; p^{\bullet r} \leftrightarrow \bullet r, \; p^{\bullet(q\vee p^{\bullet r})} \leftrightarrow \bullet(q \vee p^{\bullet r}), \\ p^{\bullet(p^{\bullet p}\vee r)} \leftrightarrow \bullet(p^{\bullet p} \vee r), \; p^{\bullet(p\wedge p^{\bullet(q\vee p^{\bullet r})})} \leftrightarrow \bullet(p \wedge p^{\bullet(q\vee p^{\bullet r})}) \end{array} \right\}$$

Finally, for any ranked model $\mathscr{R}$ on the set of propositional variables $\mathcal{P}^{\mathcal{K}}$, we define its *full $\mathcal{K}$-extended version* $\widehat{\mathscr{R}}$ as follows: Let $\mathscr{R}_0 := \mathscr{R}$, and for all $i = 1, \ldots, n$, $\mathscr{R}_i := \mathscr{R}_{i-1}^{\bullet}$. Then we let $\widehat{\mathscr{R}} := \mathscr{R}_n$.

Using Proposition 15 and Corollary 16 we then obtain the result we require.

THEOREM 18 *Let $\mathcal{K}$ be an $\mathcal{L}^{\bullet}$-knowledge base. Then*

1. *For every $\mathscr{R}$ on $\mathcal{P}^{\mathcal{K}}$, its full $\mathcal{K}$-extended version $\widehat{\mathscr{R}}$ is such that $\widehat{\mathscr{R}} \Vdash \widehat{E}$;*

2. *A ranked model $\mathscr{R}'$ on $\mathcal{P}^{\widehat{\mathcal{K}}}$ is a model of $\widehat{E}$ if and only if there is a ranked model $\mathscr{R}$ on $\mathcal{P}^{\mathcal{K}}$ such that $\mathscr{R}' = \widehat{\mathscr{R}}$;*

3. *For all $\beta \in \mathcal{L}^{\bullet}$, $\mathscr{R} \Vdash \beta$ if and only if $\widehat{\mathscr{R}} \Vdash \widehat{\beta}$;*

4. *Let $\mathscr{R}$ be a ranked model $\mathscr{R}$ on $\mathcal{P}^{\mathcal{K}}$. Then $\mathscr{R} \Vdash \mathcal{K}$ if and only if $\widehat{\mathscr{R}} \Vdash \widehat{\mathcal{K}}$.*

Proof. The proofs of Parts 1–3 follow by induction from the proofs of the corresponding Parts 1–3 of Proposition 15. The proof of Part 4 follows by induction from Corollary 16. ■

5 Belief Revision and Typicality

Given the well-known connection between propositional rational consequence relations and AGM-style belief revision [Alchourrón *et al.*, 1985], as developed by Gärdenfors and Makinson [1994], it is perhaps not surprising that propositional AGM belief revision can be expressed using the typicality operator. In this section we make this claim more precise. The formal representation of propositional AGM revision that we provide below is based on that given by Katsuno and Mendelzon [1991].

The starting point here is to fix a non-empty subset $\mathcal{V}$ of $\mathscr{U}$ (as done by Kraus et al. [1990]), and to assume that everything is done within the context of $\mathcal{V}$. In that sense, $\mathcal{V}$ becomes the set of *all* valuations that are available to us. This is slightly more general than the Katsuno-Mendelzon framework, which assumes $\mathcal{V}$ to be equal to $\mathscr{U}$, but is a special case of the original AGM approach. To reflect this restriction, we use $Mod_{\mathcal{V}}(\alpha)$ to denote the set $Mod(\alpha) \cap \mathcal{V}$. In the same vein, in the rationality postulates below, validity is understood to be modulo $\mathcal{V}$. That is, for $\alpha \in \mathcal{L}$ (i.e., α is a propositional sentence), we let $\models \alpha$ if and only if $Mod_{\mathcal{V}}(\alpha) = \mathcal{V}$.

Next, we fix a knowledge base $\kappa \in \mathcal{L}$ (i.e., represented as a propositional formula) such that $Mod_{\mathcal{V}}(\kappa) \neq \emptyset$. A revision operator $\circ$ on $\mathcal{L}$ for κ is a function from $\mathcal{L}$ to $\mathcal{L}$. Intuitively, $\kappa \circ \alpha$ is the result of revising κ by α (clearly the models of $\kappa \circ \alpha$ should be in $\mathcal{V}$). An AGM revision operator $\circ$ on $\mathcal{L}$ for κ is a revision operator on $\mathcal{L}$ for κ which satisfies the following six properties:

(R1) $\models (\kappa \circ \alpha) \rightarrow \alpha$

(R2) If $\not\models \neg(\kappa \wedge \alpha)$, then $\models (\kappa \circ \alpha) \leftrightarrow (\kappa \wedge \alpha)$

(R3) If $\not\models \neg\alpha$, then $\not\models \neg(\kappa \circ \alpha)$

(R4) If $\models \kappa_1 \leftrightarrow \kappa_2$ and $\models \alpha_1 \leftrightarrow \alpha_2$, then $\models (\kappa_1 \circ \alpha_1) \leftrightarrow (\kappa_2 \circ \alpha_2)$

(R5) $\models ((\kappa \circ \alpha) \wedge \beta) \rightarrow (\kappa \circ (\alpha \wedge \beta))$

(R6) If $\not\models \neg((\kappa \circ \alpha) \wedge \beta)$, then $\models (\kappa \circ (\alpha \wedge \beta)) \rightarrow ((\kappa \circ \alpha) \wedge \beta)$

Given $\mathcal{V}$, a ranked model $\mathscr{R} = \langle \mathcal{V}, \prec \rangle$ is defined as *κ-faithful* if and only if $\min_{\prec} \mathcal{V} = [\![\kappa]\!]$. We say that a revision operator $\circ_{\mathscr{R}}$ (on $\mathcal{L}$) is *defined by* a κ-faithful ranked model $\mathscr{R}$ if and only if $Mod_{\mathcal{V}}(\kappa \circ_{\mathscr{R}} \alpha) = \min_{\prec}[\![\alpha]\!]$. Katsuno and Mendelzon [1991] proved that, for $\mathcal{V} = \mathscr{U}$, (i) every revision operator $\circ_{\mathscr{R}}$

defined by a κ-faithful ranked model $\mathscr{R}$ is an AGM revision operator (on $\mathcal{L}$), and (*ii*) for every AGM revision operator $\circ$ (on $\mathcal{L}$) for κ, there is a κ-faithful ranked model $\mathscr{R}$ such that $Mod_{\mathcal{V}}(\kappa \circ \alpha) = Mod_{\mathcal{V}}(\kappa \circ_{\mathscr{R}} \alpha)$.

In what follows, we show that the revision operator $\circ$ can be expressed in $\mathcal{L}^{\bullet}$ using typicality. The key insight is to identify the knowledge base κ to be revised with the formula $\bullet\top$, while $\kappa \circ \alpha$ is identified with $\bullet\alpha$.

PROPOSITION 19 *Let $\kappa \in \mathcal{L}$ such that $Mod_{\mathcal{V}}(\kappa) \neq \emptyset$ and let $\mathscr{R} = \langle \mathcal{V}, \prec \rangle$ be a κ-faithful ranked model.*

1. *For every $\alpha \in \mathcal{L}$, $[\![\kappa \circ_{\mathscr{R}} \alpha]\!] = [\![\bullet\alpha]\!]$;*
2. *Let $\circ$ be any AGM revision operator (on $\mathcal{L}$) for κ. Then there is a κ-faithful ranked model $\mathscr{R} = \langle \mathcal{V}, \prec \rangle$ such that $Mod_{\mathcal{V}}(\kappa \circ \alpha) = [\![\bullet\alpha]\!]$.*

Proof. Let $\kappa \in \mathcal{L}$ and let $\mathscr{R} = \langle \mathcal{V}, \prec \rangle$ be a κ-faithful ranked model.

Proving 1: Pick any $\alpha \in \mathcal{L}$. By definition, $[\![\kappa \circ_{\mathscr{R}} \alpha]\!] = \min_{\prec}[\![\alpha]\!]$ and $[\![\bullet\alpha]\!] = \min_{\prec}[\![\alpha]\!]$, so the result holds.

Proving 2: Pick any AGM belief revision operator $\circ$ (on $\mathcal{L}$) for κ. First we need to prove that there is a κ-faithful ranked model $\mathscr{R} = \langle \mathcal{V}, \prec \rangle$ such that $Mod_{\mathcal{V}}(\kappa \circ \alpha) = \min_{\prec}[\![\alpha]\!]$. The proof is exactly the same as the proof of Theorem 3.3 provided by Katsuno and Mendelzon [1991], but with $\mathscr{U}$ replaced by $\mathcal{V}$. The result then follows from the fact that $[\![\bullet\alpha]\!] = \min_{\prec}[\![\alpha]\!]$. ■

This result shows that propositional AGM belief revision can be embedded in PTL. But we can take this a step further and extend revision to apply to the language of PTL, i.e., to $\mathcal{L}^{\bullet}$, as well. So, with a given (non-empty) $\mathcal{V}$ still fixed, and for $\alpha \in \mathcal{L}^{\bullet}$, we let $\mathcal{R}^{\alpha}_{\mathcal{V}} := \{\mathscr{R} \mid \mathscr{R} = \langle \mathcal{V}, \prec \rangle \text{ and } \min_{\prec}[\![\top]\!] = [\![\alpha]\!]\}$. Then we fix a $\kappa \in \mathcal{L}^{\bullet}$ such that $\mathcal{R}^{\kappa}_{\mathcal{V}} \neq \emptyset$. The definition of a revision operator $\circ$ is then the same as above, except that it is now with respect to $\mathcal{L}^{\bullet}$. In order to define an AGM belief revision operator on $\mathcal{L}^{\bullet}$ we need to rephrase the six revision postulates with respect to κ and the elements $\mathscr{R}$ of $\mathcal{R}^{\kappa}_{\mathcal{V}}$:

(R1$^{\bullet}$) $\mathscr{R} \Vdash (\kappa \circ \alpha) \rightarrow \alpha$

(R2$^{\bullet}$) If $\mathscr{R} \not\Vdash \neg(\kappa \wedge \alpha)$, then $\mathscr{R} \Vdash (\kappa \circ \alpha) \leftrightarrow (\kappa \wedge \alpha)$

(R3$^{\bullet}$) If $\mathscr{R} \not\Vdash \neg\alpha$, then $\mathscr{R} \not\Vdash \neg(\kappa \circ \alpha)$

(R4$^{\bullet}$) If $\mathscr{R} \Vdash \alpha_1 \leftrightarrow \alpha_2$, then $\mathscr{R} \Vdash (\kappa \circ \alpha_1) \leftrightarrow (\kappa \circ \alpha_2)$

(R5$^{\bullet}$) $\mathscr{R} \Vdash ((\kappa \circ \alpha) \wedge \beta) \rightarrow (\kappa \circ (\alpha \wedge \beta))$

(R6$^{\bullet}$) If $\mathscr{R} \not\Vdash \neg((\kappa \circ \alpha) \wedge \beta)$, then $\mathscr{R} \Vdash (\kappa \circ (\alpha \wedge \beta)) \rightarrow ((\kappa \circ \alpha) \wedge \beta)$

This gives us a representation result similar to that of Katsuno and Mendelzon in the propositional case, but with the revision operator now defined on the more expressive language $\mathcal{L}^{\bullet}$.

THEOREM 20 *Let* $\kappa \in \mathcal{L}^\bullet$ *such that* $\mathcal{R}^\kappa_{\mathcal{V}} \neq \emptyset$ *and let* $\mathscr{R} \in \mathcal{R}^\kappa_{\mathcal{V}}$. *Then* $\circ_{\mathscr{R}}$ *satisfies the postulates* $(R1^\bullet)$*–*$(R6^\bullet)$*. Conversely, let* $\circ$ *be a revision operator (for a fixed* $\mathcal{V}$ *and* κ*) satisfying the postulates* $(R1^\bullet)$*–*$(R6^\bullet)$*. Then there is an* $\mathscr{R} \in \mathcal{R}^\kappa_{\mathcal{V}}$ *such that* $Mod_{\mathcal{V}}(\kappa \circ \alpha) = Mod_{\mathcal{V}}(\kappa \circ_{\mathscr{R}} \alpha)$.

Proof. Let $\mathscr{R} = \langle \mathcal{V}, \prec \rangle \in \mathcal{R}^\kappa_{\mathcal{V}}$ be a κ-faithful ranked model. First we prove that $\circ_{\mathscr{R}}$ satisfies Postulates $(R1^\bullet)$–$(R6^\bullet)$. For $(R1^\bullet)$, observe that $[\![\kappa \circ_{\mathscr{R}} \alpha]\!] = \min_\prec [\![\alpha]\!] \subseteq [\![\alpha]\!]$. For $(R2^\bullet)$, suppose there is a $v \in \mathcal{V}$ such that $v \in [\![\kappa \wedge \alpha]\!]$. Since $\mathscr{R}$ is κ-faithful, it follows that for every $w \in [\![\kappa \wedge \alpha]\!]$, $w \in \min_\prec [\![\top]\!]$. Therefore $[\![\kappa \circ_{\mathscr{R}} \alpha]\!] = \min_\prec [\![\top]\!] = [\![\kappa \wedge \alpha]\!]$ from which it follows that $\mathscr{R} \Vdash (\kappa \circ_{\mathscr{R}} \alpha) \leftrightarrow (\kappa \wedge \alpha)$. For $(R3^\bullet)$, suppose there is a $v \in \mathcal{V}$ such that $v \in [\![\alpha]\!]$. Then $[\![\kappa \circ_{\mathscr{R}} \alpha]\!] = \min_\prec [\![\alpha]\!] \neq \emptyset$, and so $\mathscr{R} \not\Vdash \neg(\kappa \circ_{\mathscr{R}} \alpha)$. For $(R4^\bullet)$, suppose that $\mathscr{R} \Vdash \alpha_1 \leftrightarrow \alpha_2$. Then $\min_\prec [\![\alpha_1]\!] = \min_\prec [\![\alpha_2]\!]$ from which it follows that $\mathscr{R} \Vdash (\kappa \circ_{\mathscr{R}} \alpha_1) \leftrightarrow (\kappa \circ_{\mathscr{R}} \alpha_2)$. For $(R5^\bullet)$, pick any $v \in \mathcal{V}$ and suppose that $v \in [\![(\kappa \circ_{\mathscr{R}} \alpha) \wedge \beta]\!]$. That is, $v \in \min_\prec [\![\alpha]\!] \cap [\![\beta]\!]$. We need to show that $v \in \min_\prec [\![\alpha \wedge \beta]\!]$. If this is not the case, there is a $w \in \min_\prec [\![\alpha \wedge \beta]\!]$ such that $w \prec v$. But this cannot be since $v \in \min_\prec [\![\alpha]\!]$. So $(R5^\bullet)$ is satisfied. For $(R6^\bullet)$, suppose that $\mathscr{R} \not\Vdash \neg(\kappa \circ_{\mathscr{R}} \alpha) \wedge \beta$. That is, there is a $w \in \min_\prec [\![\alpha]\!] \cap [\![\beta]\!]$. Now pick a $v \in \mathcal{V}$ and suppose that $v \in [\![\kappa \circ_{\mathscr{R}} (\alpha \wedge \beta)]\!]$. That is, $v \in \min_\prec [\![\alpha \wedge \beta]\!]$. We need to show that $v \in \min_\prec ([\![\alpha]\!] \cap [\![\beta]\!])$. Since $v \in \min_\prec [\![\beta]\!]$, if this is not the case, there is an $x \in \min_\prec ([\![\alpha]\!] \cap [\![\neg\beta]\!])$ such that $x \prec v$. But then $w \prec v$ as well, which is impossible (because $v \in \min_\prec [\![\alpha \wedge \beta]\!]$). So $(R6^\bullet)$ is satisfied.

Conversely, let $\circ$ be a revision operator for κ satisfying Postulates $(R1^\bullet)$–$(R6^\bullet)$. We construct a ranked model $\mathscr{R} \in \mathcal{R}^\kappa_{\mathcal{V}}$ such that $Mod_{\mathcal{V}}(\kappa \circ \alpha) = Mod_{\mathcal{V}}(\kappa \circ_{\mathscr{R}} \alpha)$. The construction is essentially the same as that of Katsuno and Mendelzon [1991, Theorem 3.3]. We let $\mathscr{R} = \langle \mathcal{V}, \prec \rangle$, where $\prec$ is obtained as follows. Define a binary relation $\preceq$ on $\mathcal{V}$ such that for every $v, w \in \mathcal{V}$, $v \preceq w$ if and only if either $v \in [\![\kappa]\!]$ or $v \in [\![\kappa \circ f(v,w)]\!]$ where $f(v,w)$ is any element of $\mathcal{L}$ (i.e., a propositional formula) such that $[\![f(v,w)]\!] = \{v, w\}$. Katsuno and Mendelzon show that $\preceq$ is a total preorder (a binary relation that is reflexive, transitive, and connected). We let $\prec$ be the strict version of $\preceq$: $v \prec w$ if and only if $v \preceq w$ and $w \not\preceq v$. It follows immediately that $\prec$ is a modular ordering and that $\mathscr{R}$ is therefore a ranked model. To show that $\mathscr{R} \in \mathcal{R}^\kappa_{\mathcal{V}}$ we need to show that $\min_\prec [\![\top]\!] = [\![\kappa]\!]$. We can split this into two parts: (i) If $v, w \in [\![\kappa]\!]$, then $v \not\prec w$; (ii) If $v \in [\![\kappa]\!]$ and $w \notin [\![\kappa]\!]$, then $v \prec w$. Part (i) follows immediately from the definition of the relation $\preceq$. For Part (ii), suppose that $v \in [\![\kappa]\!]$ and $w \notin [\![\kappa]\!]$. From Postulate $(R2^\bullet)$ it follows that $[\![\kappa \circ f(v,w)]\!] = \{v\}$. By the definition of $\preceq$, it then follows that $v \prec w$. It remains to show that $[\![\kappa \circ \alpha]\!] = [\![\kappa \circ_{\mathscr{R}} \alpha]\!] = \min_\prec [\![\alpha]\!]$. The proof is the same as that of Katsuno and Mendelzon [1991, Theorem 3.3]. ■

6 Rational Consequence on $\mathcal{L}^\bullet$

We have seen in Section 5 that typicality can be used to express propositional AGM belief revision, as well as AGM belief revision defined for PTL. From Proposition 11 we know that rational consequence for propositional logic can be expressed in PTL itself. In this section we shall complete the picture by showing that (i) the expected connection between rational consequence relations and AGM belief revision for PTL holds, and that (*ii*) rational consequence for PTL can be expressed in PTL itself, a result analogous to Theorem 20.

As in Section 5, we start by fixing a set of valuations $\mathcal{V} \subseteq \mathscr{U}$. In this case, however, $\mathcal{V}$ is allowed to be empty as well. Let $\mathcal{R}_{\mathcal{V}} := \{\mathscr{R} \mid \mathscr{R} = \langle \mathcal{V}, \prec \rangle\}$. Then we let $\mid\!\sim$ be a binary relation on $\mathcal{L}^{\bullet}$. We say that $\mid\!\sim$ is a *rational consequence relation on* $\mathcal{L}^{\bullet}$ (with respect to $\mathscr{R} \in \mathcal{R}_{\mathcal{V}}$) if and only if $\mid\!\sim$, viewed as a *binary connective* on $\mathcal{L}^{\bullet}$, satisfies the following seven properties, adopted from the rationality properties from Section 2:

(Ref) $\mathscr{R} \Vdash \alpha \mid\!\sim \alpha$

(LLE) $\dfrac{\mathscr{R} \Vdash \alpha \leftrightarrow \beta,\ \mathscr{R} \Vdash \alpha \mid\!\sim \gamma}{\mathscr{R} \Vdash \beta \mid\!\sim \gamma}$

(And) $\dfrac{\mathscr{R} \Vdash \alpha \mid\!\sim \beta,\ \mathscr{R} \Vdash \alpha \mid\!\sim \gamma}{\mathscr{R} \Vdash \alpha \mid\!\sim \beta \wedge \gamma}$

(Or) $\dfrac{\mathscr{R} \Vdash \alpha \mid\!\sim \gamma,\ \mathscr{R} \Vdash \beta \mid\!\sim \gamma}{\mathscr{R} \Vdash \alpha \vee \beta \mid\!\sim \gamma}$

(RW) $\dfrac{\mathscr{R} \Vdash \alpha \mid\!\sim \beta,\ \mathscr{R} \Vdash \beta \to \gamma}{\mathscr{R} \Vdash \alpha \mid\!\sim \gamma}$

(CM) $\dfrac{\mathscr{R} \Vdash \alpha \mid\!\sim \beta,\ \mathscr{R} \Vdash \alpha \mid\!\sim \gamma}{\mathscr{R} \Vdash \alpha \wedge \beta \mid\!\sim \gamma}$

(RM) $\dfrac{\mathscr{R} \Vdash \alpha \mid\!\sim \beta,\ \ \mathscr{R} \not\Vdash \alpha \mid\!\sim \neg\gamma}{\mathscr{R} \Vdash \alpha \wedge \gamma \mid\!\sim \beta}$

As in Section 2, given a ranked model $\mathscr{R}$, a pair (α, β) is in the consequence relation defined by $\mathscr{R}$ (denoted as $\alpha \mid\!\sim_{\mathscr{R}} \beta$) if and only if $\min_{\prec}[\![\alpha]\!] \subseteq [\![\beta]\!]$. In this case, however, α and β are taken to be elements of $\mathcal{L}^{\bullet}$ and not just of $\mathcal{L}$.

In order for us to describe the connection between rational consequence and AGM revision for PTL, we first consider the following additional property on defeasible consequence relations:

$$\text{(Cons)} \quad \mathscr{R} \not\Vdash \top \mid\!\sim \bot$$

It is easy to see that for a ranked model $\mathscr{R} = \langle \mathcal{V}, \prec \rangle$, $\top \mid\!\sim_{\mathscr{R}} \bot$ holds if and only if $\mathcal{V} = \emptyset$. By insisting that Property (Cons) holds, we are restricting ourselves to ranked models in which $\mathcal{V} \neq \emptyset$, a restriction that is necessary to comply with Postulate (R3) for AGM belief revision (cf. Section 5). So, we consider only the case where the (fixed) set $\mathcal{V}$ is non-empty. (It is not hard to see that if $\mathcal{V} = \emptyset$, then $\mathcal{L}^{\bullet} \times \mathcal{L}^{\bullet}$ is the only rational consequence relation satisfying all the seven rationality properties above, that $\mathscr{R} = \langle \emptyset, \emptyset \rangle$ is the only ranked model, and that $\mid\!\sim_{\mathscr{R}} = \mathcal{L}^{\bullet} \times \mathcal{L}^{\bullet}$.)

Intuitively, given a rational consequence relation $\mid\!\sim$ and a belief revision operator $\circ$ for a given knowledge base κ, the idea is to (*i*) associate κ with all the βs such that $\top \mid\!\sim \beta$ holds and (*ii*) to associate the consequences of $\kappa \circ \alpha$ with all the βs such that $\alpha \mid\!\sim \beta$ holds. Such is the approach adopted by Gärdenfors and Makinson [1994] in the propositional case.

For a rational relation $\mid\!\sim$ on $\mathcal{L}^{\bullet}$, let $C^{\mid\sim} := \{\alpha \in \mathcal{L}^{\bullet} \mid \top \mid\!\sim \alpha\}$ and let $\mathcal{K}^{\mid\sim}$ be the set of all logically strongest formulae (modulo $\mathcal{R}_{\mathcal{V}}$) to be defeasibly concluded from $\top$. That is,

$$\mathcal{K}^{\mid\sim} := \{\alpha \in C^{\mid\sim} \mid \text{ for all } \beta \in C^{\mid\sim},\ \text{if } \models \beta \to \alpha,\ \text{then } \models \alpha \to \beta\},$$

where $\models$ is understood to mean validity modulo $\mathcal{R}_{\mathcal{V}}$.

The following result establishes the connection between AGM revision and rational consequence for $\mathcal{L}^\bullet$. The result is closely related to that by Gärdenfors and Makinson [1994]. (In fact, parts of the proof of Proposition 21 rely heavily on results first obtained by Gärdenfors and Makinson.)

PROPOSITION 21

1. *Let $\mid\!\sim$ be a rational consequence relation on $\mathcal{L}^\bullet$ also satisfying (Cons), and let $\kappa \in \mathcal{K}^{\mid\sim}$. Then there is a κ-faithful ranked model $\mathscr{R} \in \mathcal{R}_{\mathcal{V}}$ for which the AGM revision operator $\circ_{\mathscr{R}}$ on $\mathcal{L}^\bullet$ for κ is such that $\mathscr{R} \Vdash \alpha \mid\!\sim \beta$ if and only if $\mathscr{R} \Vdash (\kappa \circ_{\mathscr{R}} \alpha) \rightarrow \beta$.*

2. *Let κ be any element of $\mathcal{L}^\bullet$ such that $\not\models \neg\kappa$, and let $\circ$ be an AGM revision operator on $\mathcal{L}^\bullet$ for κ. Let $\mathscr{R} \in \mathcal{R}_{\mathcal{V}}^{\kappa}$ be the (κ-faithful) ranked model $\mathscr{R} \in \mathcal{R}_{\mathcal{V}}^{\kappa}$ such that $\circ = \circ_{\mathscr{R}}$.[5] Let $\mid\!\sim$ be the defeasible consequence relation on $\mathcal{L}^\bullet$ such that $\alpha \mid\!\sim \beta$ if and only if $\mathscr{R} \Vdash (\kappa \circ \alpha) \rightarrow \beta$. Then $\mid\!\sim$ is a rational consequence relation (with respect to $\mathscr{R}$) also satisfying (Cons).*

Proof.
Proving 1: Consider the following translated versions of the seven rationality properties and the (Cons) property (making use of $\mathscr{R} \Vdash \alpha \mid\!\sim \beta$ if and only if $\mathscr{R} \Vdash (\kappa \circ_{\mathscr{R}} \alpha) \rightarrow \beta$).

(Ref) $\mathscr{R} \Vdash (\kappa \circ \alpha) \rightarrow \alpha$

$$\text{(LLE)}\ \frac{\mathscr{R} \Vdash \alpha \leftrightarrow \beta,\ \mathscr{R} \Vdash (\kappa \circ \alpha) \rightarrow \gamma}{\mathscr{R} \Vdash (\kappa \circ \beta) \rightarrow \gamma}$$

$$\text{(And)}\ \frac{\mathscr{R} \Vdash (\kappa \circ \alpha) \rightarrow \beta,\ \mathscr{R} \Vdash (\kappa \circ \alpha) \rightarrow \gamma}{\mathscr{R} \Vdash (\kappa \circ \alpha) \rightarrow \beta \wedge \gamma}$$

$$\text{(Or)}\ \frac{\mathscr{R} \Vdash (\kappa \circ \alpha) \rightarrow \gamma,\ \mathscr{R} \Vdash (\kappa \circ \beta) \rightarrow \gamma}{\mathscr{R} \Vdash (\kappa \circ (\alpha \vee \beta)) \rightarrow \gamma}$$

$$\text{(RW)}\ \frac{\mathscr{R} \Vdash (\kappa \circ \alpha) \rightarrow \beta,\ \mathscr{R} \Vdash \beta \rightarrow \gamma}{\mathscr{R} \Vdash (\kappa \circ \alpha) \rightarrow \gamma}$$

$$\text{(CM)}\ \frac{\mathscr{R} \Vdash (\kappa \circ \alpha) \rightarrow \beta,\ \mathscr{R} \Vdash (\kappa \circ \alpha) \rightarrow \gamma}{\mathscr{R} \Vdash (\kappa \circ (\alpha \wedge \beta)) \rightarrow \gamma}$$

$$\text{(RM)}\ \frac{\mathscr{R} \Vdash (\kappa \circ \alpha) \rightarrow \beta,\ \ \mathscr{R} \not\Vdash (\kappa \circ \alpha) \rightarrow \neg\gamma}{\mathscr{R} \Vdash (\kappa \circ (\alpha \wedge \gamma)) \rightarrow \beta}$$

(Cons) $\mathscr{R} \not\Vdash (\kappa \circ \top) \rightarrow \bot$

We show below that any $\mathscr{R} \in \mathcal{R}_{\mathcal{V}}^{\kappa}$ satisfying these properties, also satisfies the properties $(R1^\bullet)$–$(R6^\bullet)$. From Theorem 20 the result then follows.

[5] Remember that $\mathscr{R}$ exists by Theorem 20.

($R1^\bullet$) follows directly from (Ref), and ($R4^\bullet$) follows from (LLE). For ($R5^\bullet$), observe that it is equivalent to the following property

$$\text{(Con)}\ \frac{\mathscr{R} \Vdash (\kappa \circ (\alpha \wedge \beta)) \rightarrow \gamma}{\mathscr{R} \Vdash (\kappa \circ \alpha) \wedge \beta \rightarrow \gamma}$$

which, in turn, is equivalent to (Or). For Postulate ($R6^\bullet$), observe that it is equivalent to (RM).

For ($R3^\bullet$), it can be shown that $\mid\!\sim$ satisfies the following property:

$$\text{(WRM)}\ \frac{\mathscr{R} \not\Vdash (\kappa \circ \top) \rightarrow \neg\alpha, \quad \mathscr{R} \Vdash (\kappa \circ \top) \rightarrow (\alpha \rightarrow \beta)}{\mathscr{R} \Vdash (\kappa \circ \alpha) \rightarrow \beta}$$

and that ($R3^\bullet$) follows from (WRM).

For ($R2^\bullet$), it can be shown that $\mid\!\sim$ satisfies the following property:

$$\text{(WC)}\ \frac{\mathscr{R} \Vdash (\kappa \circ \alpha) \rightarrow \beta}{\mathscr{R} \Vdash (\kappa \circ \top) \rightarrow (\alpha \rightarrow \beta)}$$

and that ($R2^\bullet$) follows from (WRC).

Proving 2: We need to show that $\mid\!\sim$ satisfies the seven rationality properties, plus (Cons). This can be done by replacing each of these properties with the translated versions in Part 1 of this proof (involving $\circ_{\mathscr{R}}$), and checking whether the revision operator $\circ_{\mathscr{R}}$ satisfies the translated properties.

Showing that (Ref), (LLE) and (RW) hold is easy. To show that (Cons) holds observe that, since $\mathscr{R}$ is a κ-faithful ranked model, it follows that $\min_\prec[\![\top]\!] \neq \emptyset$. For (And), observe that from the fact that $\min_\prec[\![\alpha]\!] \subseteq [\![\beta]\!]$ and $\min_\prec[\![\alpha]\!] \subseteq [\![\gamma]\!]$ it follows that $\min_\prec[\![\alpha]\!] \subseteq [\![\beta]\!] \cap [\![\gamma]\!]$. For (Or), observe that from the fact that $\min_\prec[\![\alpha]\!] \subseteq [\![\gamma]\!]$ and $\min_\prec[\![\beta]\!] \subseteq [\![\gamma]\!]$ it follows that $\min_\prec[\![\alpha \vee \beta]\!] \subseteq [\![\gamma]\!]$ (since $\min_\prec[\![\alpha \vee \beta]\!] \subseteq \min_\prec[\![\alpha]\!] \cup \min_\prec[\![\beta]\!]$). For (CM), suppose that $\min_\prec[\![\alpha]\!] \subseteq [\![\beta]\!]$ and $\min_\prec[\![\alpha]\!] \subseteq [\![\gamma]\!]$ and pick a $v \in \min_\prec[\![\alpha \wedge \beta]\!]$. Then it must be the case that $v \in \min_\prec[\![\alpha]\!]$ (if there were a $w \in \min_\prec[\![\alpha]\!]$ such that $w \prec v$, then it would be the case that $w \in \min_\prec[\![\alpha \wedge \beta]\!]$), from which it follows that $v \in \min_\prec[\![\gamma]\!]$. For (RM), suppose that $\min_\prec[\![\alpha]\!] \subseteq [\![\beta]\!]$ and $\min_\prec[\![\alpha]\!] \not\subseteq [\![\neg\gamma]\!]$ and pick a $v \in \min_\prec[\![\alpha \wedge \gamma]\!]$. Then it must be the case that $v \in \min_\prec[\![\alpha]\!]$ (from $\min_\prec[\![\alpha]\!] \not\subseteq [\![\neg\gamma]\!]$ we know there is at least one $w \in \min_\prec[\![\alpha]\!]$ such that $w \in [\![\gamma]\!]$), from which it follows that $v \in \min_\prec[\![\beta]\!]$. ■

Proposition 21 then allows us to obtain a representation result for rational consequence relations on $\mathcal{L}^\bullet$, as the following corollary shows.

COROLLARY 22 *For every ranked model $\mathscr{R}$, $\mid\!\sim_{\mathscr{R}}$ is a rational consequence relation on $\mathcal{L}^\bullet$. Conversely, for every rational consequence relation $\mid\!\sim$ on $\mathcal{L}^\bullet$ there exists a ranked model $\mathscr{R}$ such that $\mid\!\sim_{\mathscr{R}} = \mid\!\sim$.*

Proof. Pick a ranked model $\mathscr{R} = \langle \mathcal{V}, \prec \rangle$. If $\mathcal{V} \neq \emptyset$, then $\mathscr{R} \in \mathcal{R}_{\mathcal{V}}$. Let κ be such that $\min_{\prec}[\![\top]\!] = [\![\kappa]\!]$. So $\mathscr{R} \in \mathcal{R}^{\kappa}_{\mathcal{V}}$ and by Part 2 of Proposition 21 it follows that $\mid\!\sim_{\mathscr{R}}$ is a rational consequence relation on $\mathcal{L}^{\bullet}$. If $\mathcal{V} = \emptyset$, then $\mid\!\sim_{\mathscr{R}} = \mathcal{L}^{\bullet} \times \mathcal{L}^{\bullet}$. And it is easy to see that $\mid\!\sim$ is then a rational consequence relation.

Conversely, let $\mid\!\sim$ be a rational consequence relation on $\mathcal{L}^{\bullet}$ and let $\kappa \in \mathcal{K}^{\mid\sim}$. If $\top \not\mid\!\sim \bot$ is the case, then, from Part 1 of Proposition 21 it follows that there is an $\mathscr{R} \in \mathcal{R}^{\kappa}_{\mathcal{V}}$ such that $\mid\!\sim_{\mathscr{R}} = \mid\!\sim$. And since $\mathcal{R}^{\kappa}_{\mathcal{V}} \subseteq \mathcal{R}_{\mathcal{V}}$, the result follows. If $\top \mid\!\sim \bot$ holds, then by (RW), (LLE) and (Or), $\mid\!\sim = \mathcal{L}^{\bullet} \times \mathcal{L}^{\bullet}$. And it is easy to see that for the ranked model $\mathscr{R} = \langle \emptyset, \emptyset \rangle$, $\mid\!\sim_{\mathscr{R}} = \mid\!\sim$. ■

7 Entailment for PTL

In this section we focus on what is perhaps the central question concerning PTL from the perspective of knowledge representation and reasoning: What does it mean for a PTL formula to be *entailed* by a (finite) knowledge base $\mathcal{K}$?

Formally, we view an entailment relation as a binary relation $\models_*$ from the power set of the language under consideration (in this case $\mathcal{L}^{\bullet}$) to the language itself. Its associated *consequence* relation is defined as:

$$Cn_*(\mathcal{K}) \equiv_{\text{def}} \{\alpha \mid \mathcal{K} \models_* \alpha\}$$

Before looking at specific candidates, we propose some desired properties for such an entailment relation. The obvious place to start is to consider the properties for Tarskian consequence relations [Tarski, 1941].

(Inclusion) $\mathcal{K} \subseteq Cn_*(\mathcal{K})$

(Idempotency) $Cn_*(\mathcal{K}) = Cn_*(Cn_*(\mathcal{K}))$

(Monotonicity) If $\mathcal{K}_1 \subseteq \mathcal{K}_2$, then $Cn_*(\mathcal{K}_1) \subseteq Cn_*(\mathcal{K}_2)$

Inclusion and Idempotency are both properties we want to have satisfied, but Monotonicity is not. To see why not, it is enough to refer to the classic example from the Introduction: Let $\mathcal{K}_1 = \{\mathsf{p} \rightarrow \mathsf{b}, \bullet\mathsf{b} \rightarrow \mathsf{f}\}$ ("penguins are birds" and "birds typically fly"), and let $\mathcal{K}_2 = \mathcal{K}_1 \cup \{\bullet\mathsf{p} \rightarrow \neg\mathsf{f}\}$ (add to $\mathcal{K}_1$ that "penguins typically do not fly"). Given this, we want $\bullet\mathsf{p} \rightarrow \mathsf{f} \in Cn_*(\mathcal{K}_1)$ ("penguins typically fly" as a consequence of $\mathcal{K}_1$), but we want $\bullet\mathsf{p} \rightarrow \mathsf{f} \notin Cn_*(\mathcal{K}_2)$ ("penguins typically fly" *not* as a consequence of $\mathcal{K}_2$), thereby invalidating Monotonicity.

In addition to Inclusion and Idempotency we require $\models_*$ to behave classically when presented with propositional information only (below $\models$ denotes classical entailment):

(Classic) If $\mathcal{K} \subseteq \mathcal{L}$, then for every $\alpha \in \mathcal{L}$, $\mathcal{K} \models_* \alpha$ if and only if $\mathcal{K} \models \alpha$

Therefore, we also require that the classical consequences of a knowledge base expressed in $\mathcal{L}^{\bullet}$ be classically closed — below the $Cn(\cdot)$ operator refers to classical consequence of the propositional language $\mathcal{L}$:

(Classic Closure) $Cn_*(\mathcal{K}) \cap \mathcal{L} = Cn(Cn_*(\mathcal{K}) \cap \mathcal{L})$

We now consider four obvious candidates for the notion of entailment in PTL:

(1) $\mathcal{K} \models_1 \alpha$ if and only if for all $\mathscr{R}$ such that $\mathscr{R} \Vdash \mathcal{K}$, $\mathscr{R} \Vdash \alpha$ ('Global')

(2) $\mathcal{K} \models_2 \alpha$ if and only if for all $\mathscr{R}$, $[\![\bigwedge \mathcal{K}]\!] \subseteq [\![\alpha]\!]$ ('Local')

(3) $\mathcal{K} \models_3 \alpha$ if and only if for all $\mathscr{R}$ such that $\mathscr{R} \Vdash \bullet \bigwedge \mathcal{K}, \mathscr{R} \Vdash \alpha$ ('Classical')

(4) $\mathcal{K} \models_4 \alpha$ if and only if for all $\mathscr{R}$, $[\![\bullet \bigwedge \mathcal{K}]\!] \subseteq [\![\alpha]\!]$ ('Supra-Local')

In what follows we shall analyze each of these candidates against the aforementioned properties.

The entailment relation $\models_1$ in (1) corresponds to the standard Tarskian notion of entailment [Tarski, 1941] applied to the semantics of PTL. It also resembles the notion of *global entailment* in modal logic [Blackburn *et al.*, 2001]. It is not difficult to see that $\models_1$ satisfies Inclusion, Idempotency, Classic, and Classic Closure, properties that we want. Note, though, that $\models_1$ also satisfies Monotonicity, a property that we do *not* want.

Entailment relation $\models_2$ in (2) is the 'local' version of $\models_1$ in the modal sense and, as such, is stronger than $\models_1$. $\models_1$ does not imply $\models_2$, as shown by the following example: we have $\{\bullet(\alpha \wedge \beta)\} \models_1 \bullet\alpha$ but $\{\bullet(\alpha \wedge \beta)\} \not\models_2 \bullet\alpha$. It is not hard to see that $\models_2$ is also a Tarskian consequence relation and, as such, it satisfies the Monotonicity property as well.

The entailment relation $\models_3$ in (3) above boils down to a version of classical entailment in that only ranked models with an empty preference relation take part in its definition — remember Proposition 10. It is easy to see that $\models_3$ is monotonic and therefore insufficient according to our desiderata.

Finally, note that the option represented by $\models_4$ in (4) is weaker than $\models_2$ in (2), and therefore we have $\models_2 \subseteq \models_4$. Moreover, it is not hard to see that $\models_4$ is *non-monotonic*, which puts it in a good position as an appropriate candidate for PTL-entailment. Unfortunately, there is an argument which eliminates $\models_4$ from contention as a viable form of entailment. We explore this in more detail in what follows.

DEFINITION 23 *Let* $\mathcal{K} \subseteq \mathcal{L}^\bullet$ *and let* $\models_n$, $1 \leq n \leq 4$, *be one of the entailment relations above. Let*

$$\mid\!\sim_n^{\mathcal{K}} := \{(\alpha, \beta) \mid \alpha, \beta \in \mathcal{L},\ \mathcal{K} \models_n \bullet\alpha \rightarrow \beta\}$$

We say $\mid\!\sim_n^{\mathcal{K}}$ *is the (propositional) consequence relation generated by* $\mathcal{K}$ *and* $\models_n$.

PROPOSITION 24 *There is* $\mathcal{K} \subseteq \mathcal{L}^\bullet$ *for which* $\mid\!\sim_4^{\mathcal{K}}$ *is not a preferential consequence relation.*

Proof. Let $\mathcal{K} = \{\bullet\mathsf{b} \rightarrow \mathsf{f}, \bullet\mathsf{b} \rightarrow \mathsf{w}\}$ ("typical birds fly" and "typical birds have wings"). Let $\mathscr{R} = \langle \mathcal{V}, \prec \rangle$, where $\mathcal{V} = \{110, 100\}$ and $\prec = \{(100, 110)\}$ (100 is more preferred than 110). Then $[\![\bullet((\bullet\mathsf{b} \rightarrow \mathsf{f}) \wedge (\bullet\mathsf{b} \rightarrow \mathsf{w}))]\!] = 110$, but $110 \notin [\![\bullet(\mathsf{b} \wedge \mathsf{w}) \rightarrow \mathsf{f}]\!]$. So we have both $\mathsf{b} \mid\!\sim_4^{\mathcal{K}} \mathsf{f}$ and $\mathsf{b} \mid\!\sim_4^{\mathcal{K}} \mathsf{w}$, but $\mathsf{b} \wedge \mathsf{w} \not\mid\!\sim_4^{\mathcal{K}} \mathsf{f}$. Hence $\mid\!\sim_4^{\mathcal{K}}$ does not satisfy cautious monotonicity (CM) and is therefore not a preferential consequence relation. ∎

This result rules $\models_4$ out as an appropriate notion of entailment for PTL, as the propositional defeasible consequence relation it induces does not comply with the basic KLM properties from Section 2. The proposition above also raises the obvious question on how the other entailment relations behave under a similar scrutiny.

PROPOSITION 25

1. *For every* $\mathcal{K} \subseteq \mathcal{L}^\bullet$, $\mathrel{|\!\sim}_1^{\mathcal{K}}$ *is a preferential consequence relation;*
2. *There is* $\mathcal{K} \subseteq \mathcal{L}^\bullet$ *for which* $\mathrel{|\!\sim}_2^{\mathcal{K}}$ *is not a preferential consequence relation;*
3. *For every* $\mathcal{K} \subseteq \mathcal{L}^\bullet$, $\mathrel{|\!\sim}_3^{\mathcal{K}}$ *satisfies Monotonicity:*

$$\frac{\alpha \mathrel{|\!\sim} \beta}{\alpha \wedge \gamma \mathrel{|\!\sim} \beta}$$

Proof.
Proving Part 1: Let $\mathcal{K} \subseteq \mathcal{L}^\bullet$ and let $\mathscr{R}$ be a ranked model. Then, given $\alpha \in \mathcal{L}^\bullet$, we clearly have $\mathscr{R} \Vdash \bullet\alpha \rightarrow \alpha$, so (Ref) holds. Let $\alpha, \beta, \gamma \in \mathcal{L}$ and suppose that $\mathscr{R} \Vdash \bullet\alpha \rightarrow \beta$ and $\models \beta \rightarrow \gamma$ ($\models$ here refers to classical validity). Then $\mathscr{R} \Vdash \bullet\alpha \rightarrow \gamma$ and then (RW) holds. Now suppose that $\mathscr{R} \Vdash \bullet\alpha \rightarrow \gamma$ and $\models \alpha \leftrightarrow \beta$. Then $\bullet\beta \rightarrow \gamma$ so (LLE) also holds. Suppose $\mathscr{R} \Vdash \bullet\alpha \rightarrow \beta$ and $\mathscr{R} \Vdash \bullet\alpha \rightarrow \gamma$. Then clearly $\mathscr{R} \Vdash \bullet\alpha \rightarrow \beta \wedge \gamma$ and then (And) holds. Suppose $\mathscr{R} \Vdash \bullet\alpha \rightarrow \gamma$ and $\mathscr{R} \Vdash \bullet\beta \rightarrow \gamma$. It is not hard to check that $\mathscr{R} \Vdash \bullet(\alpha \vee \beta) \rightarrow \gamma$, so (Or) also holds. Now suppose $\mathscr{R} \Vdash \bullet\alpha \rightarrow \beta$ and $\mathscr{R} \Vdash \bullet\alpha \rightarrow \gamma$. Pick any $w \in \min_{\prec}[\![\alpha \wedge \gamma]\!]$, i.e., w is a "best $\alpha \wedge \gamma$-world". If $w \in \min_{\prec}[\![\alpha]\!]$, then $w \in [\![\beta]\!]$ (since $\mathscr{R} \Vdash \bullet\alpha \rightarrow \beta$) and so $w \in [\![\bullet(\alpha \wedge \gamma) \rightarrow \beta]\!]$. If $w \notin \min_{\prec}[\![\alpha]\!]$, then there is w' such that $w' \prec w$ and $w' \in \min_{\prec}[\![\alpha]\!]$. But then $w' \in [\![\neg\gamma]\!]$, since $w \in \min_{\prec}[\![\alpha \wedge \gamma]\!]$. So $\mathscr{R} \not\Vdash \bullet\alpha \rightarrow \gamma$, which is a contradiction. Hence $\mathscr{R} \Vdash \bullet(\alpha \wedge \gamma) \rightarrow \beta$, and therefore (CM) holds.

Proving Part 2: Let $\mathcal{K}$ and $\mathscr{R}$ be as in the proof of Proposition 24. Then $110 \in [\![\bigwedge \mathcal{K}]\!]$ (since $110 \notin \min_{\prec}[\![\mathsf{b}]\!]$). However, $110 \notin [\![\bullet(\mathsf{b} \wedge \mathsf{w}) \rightarrow \mathsf{f}]\!]$. From this it is easy to see that $\mathrel{|\!\sim}_2^{\mathcal{K}}$ does not satisfy (CM).

Proving Part 3: Let $\mathcal{K} \subseteq \mathcal{L}^\bullet$ and pick any $\mathscr{R} = \langle \mathcal{V}, \prec \rangle$ such that $\mathscr{R} \Vdash \bullet \bigwedge \mathcal{K}$. Then, by Proposition 10, we have $\min_{\prec}[\![\bigwedge \mathcal{K}]\!] = [\![\bigwedge \mathcal{K}]\!]$ and $\prec = \emptyset$. Now suppose $\mathscr{R} \Vdash \bullet\alpha \rightarrow \beta$. This means that for every $w \in \mathcal{V}$, if $w \in [\![\alpha]\!]$, then $w \in [\![\beta]\!]$. Now pick any $w \in [\![\alpha \wedge \gamma]\!]$. Since $w \in [\![\alpha]\!]$, we also have $w \in [\![\beta]\!]$. The same argument holds for any $w \in \min_{\prec}[\![\alpha \wedge \gamma]\!]$ (since $\prec = \emptyset$), and therefore we have $\mathscr{R} \Vdash \bullet(\alpha \wedge \gamma) \rightarrow \beta$. ■

From the results above we can conclude that even though none of $\models_i$, $1 \leq i \leq 4$, is an appropriate notion of entailment in the context of PTL, $\models_1$ turns out to be the best of all options in that it delivers a consequence relation that is preferential. There is, however, an additional argument against the use of $\models_1$ as well, one that is based on an adaptation of a result obtained by Lehmann and Magidor [1992] in the propositional case. To make the argument, we first present a result showing that all formulae of $\mathcal{L}^\bullet$ can be rewritten as statements of rational consequence:

LEMMA 26 *For every* $\mathscr{R}$ *and* $\alpha \in \mathcal{L}^\bullet$, $\mathscr{R} \Vdash \alpha$ *if and only if* $\mathscr{R} \Vdash \bullet\neg\alpha \rightarrow \bot$ *if and only if* $\neg\alpha \mathrel{|\!\sim}_{\mathscr{R}} \bot$. *Conversely for every* $\mathscr{R}$ *and* $\alpha, \beta \in \mathcal{L}^\bullet$, $\alpha \mathrel{|\!\sim}_{\mathscr{R}} \beta$ *if and only if* $\mathscr{R} \Vdash \bullet\alpha \rightarrow \beta$.

Proof. Let $\mathscr{R} = \langle \mathcal{V}, \prec \rangle$ and $\alpha \in \mathcal{L}^\bullet$. $\mathscr{R} \Vdash \alpha$ if and only if $[\![\alpha]\!] = \mathcal{V}$ if and only if $[\![\neg\alpha]\!] = \emptyset$ if and only if (i) $[\![\bullet\neg\alpha]\!] = \emptyset$ if and only if $[\![\neg \bullet \neg\alpha]\!] = \mathcal{V}$ if and only if $[\![\bullet\alpha \rightarrow \bot]\!] = \mathcal{V}$ if and only if $\mathscr{R} \Vdash \bullet\neg\alpha \rightarrow \bot$. But (i) is the case if and only if $\min_\prec [\![\neg\alpha]\!] = \emptyset$ if and only if $\min_\prec [\![\neg\alpha]\!] \subseteq [\![\bot]\!]$ if and only if $\neg\alpha \mathrel{|\!\sim}_{\mathscr{R}} \bot$.

The proof of the converse part is analogous to that of Proposition 11. ■

We can therefore think of $\mathcal{L}^\bullet$ as a language for expressing defeasible consequence on $\mathcal{L}^\bullet$ with $\mathrel{|\!\sim}$ viewed as the only main connective. More precisely, let $\mathcal{L}^\bullet_{|\!\sim} := \{\alpha \mathrel{|\!\sim} \beta \mid \alpha, \beta \in \mathcal{L}^\bullet\}$, and for any ranked model $\mathscr{R}$, let $\mathscr{R} \Vdash \alpha \mathrel{|\!\sim} \beta$ if and only if $\alpha \mathrel{|\!\sim}_{\mathscr{R}} \beta$. The next result shows that the languages $\mathcal{L}^\bullet$ and $\mathcal{L}^\bullet_{|\!\sim}$ are equally expressive.

PROPOSITION 27 *For every* $\mathscr{R}$ *and* $\alpha \mathrel{|\!\sim} \beta \in \mathcal{L}^\bullet_{|\!\sim}$, $\mathscr{R} \Vdash \alpha \mathrel{|\!\sim} \beta$ *if and only if* $\mathscr{R} \Vdash \bullet\alpha \rightarrow \beta$. *Conversely, for every* $\mathscr{R}$ *and* $\alpha \in \mathcal{L}^\bullet$, $\mathscr{R} \Vdash \alpha$ *if and only if* $\mathscr{R} \Vdash \neg\alpha \mathrel{|\!\sim} \bot$.

Proof. Straightforward, by applying Lemma 26. ■

$\mathcal{L}^\bullet_{|\!\sim}$ is similar to the language for conditional knowledge bases studied by Lehmann and Magidor [1992], but with the propositional component replaced by $\mathcal{L}^\bullet$ (i.e., $\mathrel{|\!\sim} \subseteq \mathcal{L}^\bullet \times \mathcal{L}^\bullet$).

Based on this, we restate entailment in terms of the language $\mathcal{L}^\bullet_{|\!\sim}$, and propose an additional property that any appropriate notion of entailment ought to satisfy. Let $\mathcal{K}$ be a (finite) subset of $\mathcal{L}^\bullet_{|\!\sim}$, let $\models_*$ be a (potential) entailment relation from $\mathscr{P}(\mathcal{L}^\bullet_{|\!\sim})$ to $\mathcal{L}^\bullet_{|\!\sim}$, and let $\mathrel{|\!\sim}^{\mathcal{K}}_*$ be a defeasible consequence relation on $\mathcal{L}^\bullet$ obtained from $\models_*$ as follows: $\alpha \mathrel{|\!\sim}^{\mathcal{K}}_* \beta$ if and only if $\mathcal{K} \models_* \alpha \mathrel{|\!\sim} \beta$.

(Rationality) For every finite $\mathcal{K} \subseteq \mathcal{L}^\bullet_{|\!\sim}$, the consequence relation $\mathrel{|\!\sim}^{\mathcal{K}}_*$ obtained from $\models_*$ should be *rational*

Rationality is essentially the property for the entailment of propositional conditional knowledge bases proposed by Lehmann and Magidor [1992], but applied to the language of $\mathcal{L}^\bullet_{|\!\sim}$ (cf. Section 2). Based on their results, it follows that the consequence relation $\mathrel{|\!\sim}^{\mathcal{K}}_1$ obtained from $\models_1$ does not satisfy Rationality. In fact, analogous to one of their results [Lehmann and Magidor, 1992, Section 4.2], we have the following one.

PROPOSITION 28 *For finite* $\mathcal{K} \subseteq \mathcal{L}^\bullet_{|\!\sim}$, *let* $\mathrel{|\!\sim}^{\mathcal{K}} = \{(\alpha, \beta) \mid \alpha \mathrel{|\!\sim} \beta \in \mathcal{K}\}$, *and let* $\mathrel{|\!\sim}^P$ *be the intersection of all preferential consequence relations on* $\mathcal{L}^\bullet$ *containing* $\mathrel{|\!\sim}^{\mathcal{K}}$.[6] *For the consequence relation* $\mathrel{|\!\sim}^{\mathcal{K}}_1$ *obtained from* $\models_1$, *it follows that* $\mathrel{|\!\sim}^{\mathcal{K}}_1 = \mathrel{|\!\sim}^P$ *is a preferential consequence relation, but not necessarily a rational consequence relation.*

[6]Recall that a preferential consequence relation is one satisfying the first six properties discussed in Section 2.

Proof. The proof is analogous to that given by Lehmann and Magidor [1992, Section 4.2] in the propositional case. ■

Since $\mathcal{L}^\bullet$ and $\mathcal{L}^\bullet_{\mid\sim}$ are equally expressive (cf. Proposition 27), Proposition 28 provides additional evidence that $\models_1$ is not an appropriate form of entailment.

Having shown that none of the obvious notions of entailment above is an appropriate form of entailment for PTL, we now turn our attention to an alternative proposal. It is the notion of the *rational closure* of a conditional knowledge base, proposed by Lehmann and Magidor [1992] for the propositional case, but here applied to the language of $\mathcal{L}^\bullet_{\mid\sim}$.

DEFINITION 29 *Let* $\mid\sim_0$ *and* $\mid\sim_1$ *be rational consequence relations.* $\mid\sim_0$ *is preferable to* $\mid\sim_1$ *(written* $\mid\sim_0 \ll \mid\sim_1$*) if and only if*

- *there is an* $\alpha \mid\sim \beta \in \mid\sim_1 \setminus \mid\sim_0$ *such that for all* γ *such that* $\gamma \vee \alpha \mid\sim_0 \neg\alpha$ *and for all* δ *such that* $\gamma \mid\sim_0 \delta$*, we also have* $\gamma \mid\sim_1 \delta$*;*
- *for every* $\gamma, \delta \in \mathcal{L}$*, if* $\gamma \mid\sim \delta$ *is in* $\mid\sim_0 \setminus \mid\sim_1$*, then there is an assertion* $\rho \mid\sim \nu$ *in* $\mid\sim_1 \setminus \mid\sim_0$ *such that* $\rho \vee \gamma \mid\sim_1 \neg\gamma$*.*

The motivation for $\ll$ here is essentially that for the same ordering for the propositional case provided by Lehmann and Magidor [1992]. Given $\mathcal{K} \subseteq \mathcal{L}^\bullet_{\mid\sim}$, the idea is now to define the rational closure as the most preferred (with respect to $\ll$) of all those rational consequence relations which include $\mathcal{K}$.

LEMMA 30 *Let* $\mathcal{K} \subseteq \mathcal{L}^\bullet_{\mid\sim}$ *be finite, and let* $\mid\sim^{\mathcal{K}} := \{(\alpha, \beta) \mid \alpha \mid\sim \beta \in \mathcal{K}\}$*. Then there is a unique rational consequence relation containing* $\mid\sim^{\mathcal{K}}$ *which is preferable (with respect to* $\ll$*) to all other rational consequence relations containing* $\mid\sim^{\mathcal{K}}$*.*

This allows us to define the rational closure $\models_{rc}$ of a knowledge base on $\mathcal{L}^\bullet_{\mid\sim}$.

DEFINITION 31 *For finite* $\mathcal{K} \subseteq \mathcal{L}^\bullet_{\mid\sim}$*, let* $\mid\sim^{\mathcal{K}} := \{(\alpha, \beta) \mid \alpha \mid\sim \beta \in \mathcal{K}\}$*, and let* $\mid\sim^{\mathcal{K}}_{rc}$ *be the (unique) rational consequence relation containing* $\mid\sim^{\mathcal{K}}$ *which is preferable (with respect to* $\ll$*) to all other rational consequence relations containing* $\mid\sim^{\mathcal{K}}$*. Then* $\alpha \mid\sim \beta$ *is in the rational closure of* $\mathcal{K}$ *(written as* $\mathcal{K} \models_{rc} \alpha \mid\sim \beta$*) if and only if* $\alpha \mid\sim^{\mathcal{K}}_{rc} \beta$*.*

Definition 31 gives us a notion of rational closure for $\mathcal{L}^\bullet_{\mid\sim}$. Since $\mathcal{L}^\bullet$ and $\mathcal{L}^\bullet_{\mid\sim}$ are equally expressive (remember Proposition 27), we can use Definition 31 to define rational closure for $\mathcal{L}^\bullet$ as well:

DEFINITION 32 *Let* $\mathcal{K} \subseteq \mathcal{L}^\bullet$*,* $\alpha \in \mathcal{L}^\bullet$*, and let* $\mathcal{K}^{\mid\sim} := \{\neg\beta \mid\sim \bot \mid \beta \in \mathcal{K}\}$*. Then* α *is in the rational closure of* $\mathcal{K}$ *(written as* $\mathcal{K} \models_{rc} \alpha$*) if and only if* $\neg\alpha \mid\sim \bot$ *is in the rational closure of* $\mathcal{K}^{\mid\sim}$*.*

It is not hard to show that rational closure satisfies Inclusion, Idempotency, Classic, Classic Closure, and Rationality, but not Monotonicity. It is therefore a reasonable candidate for entailment for PTL.

We conclude this section with an outlook on another proposal for entailment for $\mathcal{L}^\bullet$ based on a semantic construction. It is inspired by a proposal by Giordano et al. [2012]. The idea is to define a partial order on a certain subclass of ranked models satisfying a knowledge base $\mathcal{K} \subseteq \mathcal{L}^\bullet$, with models lower down in the ordering being viewed as more 'conservative', in the sense that one can draw fewer conclusions from them, and therefore being more preferred.

For $\mathcal{K} \subseteq \mathcal{L}^\bullet$, let $\mathcal{V}^\mathcal{K}$ be the elements of $\mathscr{U}$ 'permitted' by $\mathcal{K}$:

$$\mathcal{V}^\mathcal{K} := \{v \mid v \in \mathcal{V} \text{ for some } \mathscr{R} = \langle \mathcal{V}, \prec \rangle \text{ such that } \mathscr{R} \Vdash \mathcal{K}\}$$

Moreover, let $\mathcal{R}^\mathcal{K} := \{\mathscr{R} = \langle \mathcal{V}^\mathcal{K}, \prec \rangle \mid \mathscr{R} \Vdash \mathcal{K}\}$. Now, for any $\mathscr{R} = \langle \mathcal{V}^\mathcal{K}, \prec \rangle \in \mathcal{R}^\mathcal{K}$, let $\mathcal{V}_0^\mathscr{R} := \min_\prec \mathcal{V}^\mathcal{K}$, and for $i > 0$ let $\mathcal{V}_i^\mathscr{R} := \min_\prec \left(\mathcal{V}^\mathcal{K} \setminus (\cup_{j=0}^{j=i-1} \mathcal{V}_j^\mathscr{R})\right)$. So $\mathcal{V}_0^\mathscr{R}$ contains the elements of $\mathcal{V}^\mathcal{K}$ lowest down with respect to $\prec$, $\mathcal{V}_1^\mathscr{R}$ contains the elements of $\mathcal{V}^\mathcal{K}$ just above $\mathcal{V}_0^\mathscr{R}$ with respect to $\prec$, etc. Next, for every $v \in \mathcal{V}^\mathcal{K}$ we define the *height* of v in $\mathscr{R}$ as $h^\mathscr{R}(v) = i$ if and only if $v \in \mathcal{V}_i^\mathscr{R}$. And based on that, we define the partial order $\preceq$ on $\mathcal{R}^\mathcal{K}$ as follows: $\mathscr{R}_1 \preceq \mathscr{R}_2$ if and only if for every $v \in \mathcal{V}^\mathcal{K}$, $h^{\mathscr{R}_1}(v) \leq h^{\mathscr{R}_2}(v)$. From this we get the following result which is a special case of Theorem 2 in the recent paper by Giordano et al. [2012].

PROPOSITION 33 (Giordano et al. [2012]) *For every $\mathcal{K} \subseteq \mathcal{L}^\bullet$, the partial order $\preceq$ on the elements of $\mathcal{R}^\mathcal{K}$ has a unique minimum element.*

This allows us to provide a definition for the notion of *minimum entailment* of a PTL knowledge base.

DEFINITION 34 *Let $\mathcal{K} \subseteq \mathcal{L}^\bullet$, $\alpha \in \mathcal{L}^\bullet$, and $\mathscr{R}^\mathcal{K}$ be the (unique) minimum element of $\mathcal{R}^\mathcal{K}$ with respect to the partial order $\preceq$ on $\mathcal{R}^\mathcal{K}$. Then α is in the minimum entailment of $\mathcal{K}$ ($\mathcal{K} \models_{\min} \alpha$) if and only if $\mathscr{R}^\mathcal{K} \Vdash \alpha$.*

It can be shown that minimum entailment as defined above satisfies Inclusion, Idempotency, Classic, Classic Closure, and Rationality, but not Monotonicity. As for rational closure, it is a reasonable candidate for entailment for $\mathcal{L}^\bullet$. In fact, the connection between rational closure and minimal entailment may even be closer than that. There is strong evidence to support the conjecture that they actually coincide, but this remains to be investigated.

8 Discussion and Related Work

To the best of our knowledge, the first attempt to formalize a notion of typicality in the context of defeasible reasoning was that by Delgrande [1987]. Given the relationship between our constructions and those by Kraus and colleagues, most of the remarks in the comparison made by Lehmann and Magidor [1992, Section 3.7] are applicable in comparing Delgrande's approach to ours and therefore we do not repeat them here.

Crocco and Lamarre [1992] as well as Boutilier [1994] have explored the links between defeasible consequence relations and notions of normality similar to the one we investigate here. In particular, Boutilier defines a family of conditional logics of normality in which a statement of the form "if α, then normally β" is formalized via a binary modality $\Rightarrow$ as a conditional $\alpha \Rightarrow \beta$. Here we achieve the same with a unary operator.

Roughly speaking, Boutilier's semantic intuition is the same as that of KLM (and therefore same as ours). The main difference is that Boutilier defines a conditional connective $\Rightarrow$ in the language, whereas Kraus et al. define $\mid\!\sim$ at a meta-level to the language. In this respect, Boutilier's approach is more general in that it allows for nested conditionals. If these are omitted, i.e., if one works in the 'flat' conditional logic in which $\Rightarrow$ is the main connective and no nesting is allowed, then one gets the same results for both preferential and rational entailment with both systems. So Boutilier achieves with modalities (he works in a bi-modal language) what Kraus and colleagues achieve with a (meta-level) preference order.

It turns out that in Boutilier's approach one cannot always capture the notion of "most typical α's" (but for a different reason than that given in Proposition 13). In Boutilier's modal logic, such a set (of *most* normal α-worlds) need not exist in general. This is because Boutilier drops the smoothness condition [Boutilier, 1994, p. 103] and therefore at any point in a ranked model one can have infinitely descending chains of more and more normal α-worlds. If one imposes smoothness in Boutilier's approach, which can be done by e.g. requiring the ordering determined by Boutilier's $\Box$ also to be *Noetherian*[7], one could then define his conditional $\Rightarrow$ more elegantly as follows:

(5) $\alpha \Rightarrow \beta \equiv_{\text{def}} \bullet\alpha \rightarrow \beta$

where, in Boutilier's notation, $\bullet\alpha$ would be given by

(6) $\bullet\alpha \equiv_{\text{def}} \alpha \wedge \Box\neg\alpha$

(Of course negated conditionals of the form $\alpha \not\Rightarrow \beta$ can then be expressed as $\neg(\bullet\alpha \rightarrow \beta)$.) In adopting smoothness and defining conditionals in this way one would expect both approaches to become equivalent modulo the underlying language — ours is propositional, whereas Boutilier's is modal. However, our statement $\bullet\alpha \rightarrow \beta$ differs from Boutilier's $\alpha \Rightarrow \beta$ in a significant way. In Boutilier's approach, a statement of the form $\alpha \Rightarrow \beta$ is true at some world (in a ranked model) if and only if it is true at *all* worlds in that ranked model [Boutilier, 1994, p. 114]. On the other hand, it is not hard to find a ranked model in which $\bullet\alpha \rightarrow \beta$ holds at a world without being true in the whole model. This establishes Boutilier's conditional as a 'global' statement, while ours has the (more general) 'local flavor'. We can easily simulate Boutilier's notion of *acceptance* [Boutilier, 1994, p. 115] by stating $\top \rightarrow (\bullet\alpha \rightarrow \beta)$.

It is also worth mentioning that our interpretation of the conditional $\Rightarrow$ in (5) above and Boutilier's differ in another subtle way, which also relates to whether one adopts smoothness or not. In (5), $\alpha \Rightarrow \beta$ is defined as "the normal α's are β's", whereas, strictly speaking, Boutilier's definition of $\alpha \Rightarrow \beta$ reads as "there is a point from which $\alpha \rightarrow \beta$ is not violated". Such a 'frontier' for

[7] By doing so Boutilier's framework becomes very close to Britz et al.'s [2009].

normality, implicitly referred to in Boutilier's definition of $\alpha \Rightarrow \beta$, is not as crisp as ours in the sense that the point where one draws the normality line might be too 'far away' (in the ordering) from the more and more normal α-worlds. One can definitely make a case for dropping the smoothness condition, but requiring it is a small price to pay compared to the much simpler account of typicality one gets and that we have investigated in this chapter.

In a description logic setting, Giordano et al. [2009b] also study notions of typicality. Semantically, they do so by placing an (absolute) ordering on *objects* in first-order domains in order to define versions of defeasible subsumption relations in the description logic $\mathcal{ALC}$. The authors moreover extend the language of $\mathcal{ALC}$ with an explicit typicality operator $\mathbf{T}$ of which the intended meaning is to single out instances of a concept that are deemed as 'typical'. That is, given an $\mathcal{ALC}$ concept C, $\mathbf{T}(C)$ denotes the most typical individuals having the property of being C in a particular DL interpretation.

Giordano et al.'s approach defines rational versions of the DL subsumption relation $\sqsubseteq$ satisfying the corresponding rationality properties stated in DL terms. Nevertheless, they do not provide representation results *à la* KLM and do not address the links with belief revision either. Recently Britz et al. [2011b; 2013b] have provided such representation results in the DL case. Even though here we have investigated typicality in a propositional setting, we expect that our representation result and constructions for the rational closure (as well as the links with belief revision) can be lifted to the DL case, thereby filling the mentioned gaps in Giordano et al.'s approach and shedding some light on the issues related to typicality and defeasible reasoning in more expressive logics.

Britz et al. [2009] investigate another embedding of propositional preferential reasoning in modal logic. In their setting, the modular ordering is an accessibility relation on possible worlds, axiomatized via a modal operator $\Box$. Without getting into the technical details of the axiomatization of their underlying modal logic, other than mentioning that their accessibility relation is a modular ordering, it is worth noting that our typicality operator can be defined in terms of their modality as $\bullet\alpha \equiv_{\text{def}} \Box\neg\alpha \wedge \alpha$ (just as the alternative formulation of Boutilier's approach in (6) above). The modal sentence $\Box\neg\alpha \wedge \alpha$ says that the worlds satisfying it are α-worlds and whatever world is more preferable than these is a $\neg\alpha$-world. In other words, these are the minimal α-worlds. The general case of defining Britz et al.'s modality in terms of our typicality operator is not possible, but in a finitely generated language as we consider here, the logics become identical.

In our enriched language the preference relation is not explicit in the syntax. The meaning of the typicality operator is *informed* by the preference relation, but the latter remains nevertheless tacit. This stands in contrast to the approaches of Baltag and Smets [2008], Boutilier [1994], Britz et al. [2009] and Giordano et al. [2009a], which cast the preference relation as an (explicit) extra modality in the language. From a knowledge representation perspective, our approach has the advantage of hiding some complex aspects of the semantics from the user (e.g. a knowledge engineer who will write down sentences in an agents knowledge base).

Finally, Britz and Varzinczak [2012; 2013] investigate another, complementary aspect of defeasibility by introducing (non-standard) modal operators allowing us to talk about relative normality in accessible worlds. With their defeasible versions of modalities, namely $\boxtimes$ and $\diamond$, formalizing respectively the notions of *defeasible necessity* and *distinct possibility*, it becomes possible to make statements of the form "α holds in all of the normal (typical) accessible worlds", thereby capturing defeasibility of what is 'expected' in target worlds. (Note that this is different from stating something like $\Box \bullet \alpha$, which says that all accessible worlds are typical α-worlds.) Such preferential versions of modalities allow for the definition of a family of modal logics in which defeasible modes of inference such as defeasible actions, knowledge and obligations can be expressed. These can be integrated either with existing $\mid\!\sim$-based modal logics [Britz *et al.*, 2011a; Britz *et al.*, 2012] or with a modal extension of our typicality operator in striving towards a coherent theory of defeasible reasoning in more expressive languages.

9 Concluding Remarks

The main contributions of the work reported in the present chapter can be summarized as follows:

- We present the logic PTL which provides a formal account of typicality in a propositional language allowing us to refer directly and concisely to the most typical situations in which a given formula holds;
- We show that we can embed the (propositional) KLM framework within the more expressive language of PTL, and we also define rational consequence relations on the language of PTL itself;
- We establish a connection between rational consequence and belief revision, both on PTL, and
- We investigate appropriate notions of entailment for PTL and propose two candidates.

For future work we are interested in algorithms for computing the appropriate forms of entailment for PTL, specifically algorithms that can be reduced to validity checking for PTL. It follows indirectly from results by Lehmann and Magidor [1992] that this type of entailment has the same worst-case complexity of validity checking for PTL. Given the aforementioned links with modal logic, we know that this is at least a PSPACE-complete problem.

As briefly alluded to above, we also plan to extend PTL to more expressive logics such as description logics and modal logics. With the introduction of a typicality operator in these languages, it becomes possible to extend the propositional properties for rational consequence and obtain a characterization that reflects the additional structure of these languages, in the syntax and, more importantly, in the semantics as well.

Finally, from a knowledge representation and reasoning perspective, when dealing with knowledge bases, issues related to modularization [Cuenca Grau *et al.*, 2006; Herzig and Varzinczak, 2005b; Herzig and Varzinczak, 2006], consistency checking [Herzig and Varzinczak, 2004; Herzig and Varzinczak, 2005a; Lang *et al.*, 2003; Zhang *et al.*, 2002], knowledge base integration [Meyer *et al.*, 2005] and maintenance [Herzig *et al.*, 2006; Varzinczak, 2008; Varzinczak, 2010] as well as versioning [Franconi *et al.*, 2010; Noy and Musen, 2002] show up. These are tasks acknowledged as important by the community in the classical case [Herzig and Varzinczak, 2007; Konev *et al.*, 2008; Moodley, 2011; Thielscher, 2011; Varzinczak, 2006] and that also make sense in a nonmonotonic setting. When moving to a defeasible approach, though, such tasks have to be reassessed and specific methods and techniques redesigned. This constitutes an avenue worthy of exploration.

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