A Relaxation of a Semiring Constraint Satisfaction Problem using Combined Semirings

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Abstract. The Semiring Constraint Satisfaction Problem (SCSP) framework is a popular approach for the representation of partial constraint satisfaction problems. In this framework preferences (semiring values) can be associated with tuples of values of the variable domains. Bistarelli et al. [1] define an abstract solution to a SCSP which consists of the best set of solution tuples for the variables in the problem. Sometimes this abstract solution may not be good enough, and in this case we want to change the constraints so that we solve a problem that is slightly different from the original problem but has an acceptable solution. In [2] we propose a relaxation of a SCSP where we define a measure of distance (a semiring value from a second semiring) between the original SCSP and a relaxed SCSP. In this paper we show how the two semirings can be combined into a single semiring. This combined semiring structure will allow us to use existing tools for SCSPs to solve Combined Semiring Relaxations of SCSPs. At this stage our work is preliminary and needs further investigation to develop into a useful algorithm.

1 Introduction

The considerable interest in *over-constrained problems*, *partial constraint satisfaction problems* and *soft constraints* is motivated by the observation that with most real-life problems, it is difficult to offer *a priori* guarantees that the input set of constraints to a constraint solver is solvable. Many real-life problems are inherently over-constrained. In order to solve an over-constrained problem we have to identify appropriate *relaxations* of the original problem that are solvable. Early approaches to such relaxations largely focussed on finding maximal subsets (with respect to set cardinality) of the original set of constraints that are solvable (such as Freuder and Wallace's work on the MaxCSP problem [3]). Subsequent efforts considered more fine-grained notions of relaxation, where entire constraints did not have to be removed from consideration ([4], [5], [6]).

Bistarelli et al. [1] proposed an abstract semiring CSP scheme that generalised most of these earlier attempts, while making it possible to define several useful new instances of the scheme. The SCSP scheme assumes the existence of a semiring of abstract preference values, such that the associated multiplicative operator is used for combining preference values, while the associated additive operator is used for comparing preference values. An SCSP constraint assigns a preference value to all possible value assignments to the variables in its signature. These preferences implicitly define a relaxation strategy.

In our previous paper [2] we define how an SCSP may be relaxed by introducing a mechanism by which we can minimally alter (or relax) constraints of the problem. We also introduce a measure of distance between an original constraint and its relaxed version that is modeled via a second semiring.

Our aim in this paper is to show that we can combine the first semiring (these semiring values are used as preference values associated with tuples of constraints) and the second semiring (these semiring values are used as distance values between constraints and relaxed constraints) into a single semiring. With a single semiring we can resort to exisiting SCSP tools to solve Relaxed SCSPs.

2 The SCSP Framework and Relaxations of SCSPs

This section contains a summary of the SCSP framework of Bistarelli et al. [1], as well as a summary of the results in [2] where we propose a technique to relax the constraints of the original problem.

Definition 1. *A c-semiring is a tuple $S = \langle A, +, \times, \mathbf{0}, \mathbf{1} \rangle$ such that*

- *A is a set with $\mathbf{0}, \mathbf{1} \in A$;*
- *$+$ is defined over (possibly infinite) sets of elements of A as follows: for all $a \in A$, $\sum(\{a\}) = a$, $\sum(\emptyset) = \mathbf{0}$ and $\sum(A) = \mathbf{1}$, and $\sum(\bigcup A_i, i \in I) = \sum(\{\sum(A_i), i \in I\})$ for all sets of indices I. (When $+$ is applied to sets of elements, we use the symbol $\sum$.)*
- *$\times$ is a commutative, associative, and binary operation such that $\mathbf{1}$ is its unit element and $\mathbf{0}$ is its absorbing element, and $\times$ distributes over $+$.*

The elements of the set A are the preference values to be assigned to tuples of values of the domains of constraints. Let $\leqslant_S$ be a partial order over A: $\alpha \leqslant_S \beta$ iff $\alpha + \beta = \beta$. $\mathbf{0}$ is the minimum element and $\mathbf{1}$ is the maximum element.

Definition 2. *Consider a constraint system $CS = \langle S_p, D, V \rangle$ where $S_p = \langle A_p, +_p, \times_p, \mathbf{0}_p, \mathbf{1}_p \rangle$ is a c-semiring, V is an ordered finite set of variables, and D is a finite set of allowed values for the variables in V. A constraint over CS is a pair $c = \langle def_c^p, con_c \rangle$ with $con_c \subseteq V$, and $def_c^p : D^k \rightarrow A_p$ (k is the cardinality of con_c). A Semiring Constraint Satisfaction Problem (SCSP) over CS is a pair $P = \langle C, con \rangle$ where C is a finite set of constraints over CS and $con = \bigcup_{c \in C} con_c$. $\langle def_{c_1}^p, con_c \rangle \in C$ and $\langle def_{c_2}^p, con_c \rangle \in C$ implies $def_{c_1}^p = def_{c_2}^p$.*

Definition 3. *Given a constraint system $CS = \langle S_p, D, V \rangle$ where $S_p = \langle A_p, +_p, \times_p, \mathbf{0}_p, \mathbf{1}_p \rangle$, and two constraints $c_1 = \langle def_{c_1}^p, con_{c_1} \rangle$ and $c_2 = \langle def_{c_2}^p, con_{c_2} \rangle$ over CS, their combination, $c_1 \otimes c_2$, is the constraint $c = \langle def_c^p, con_c \rangle$ with $con_c = con_{c_1} \cup con_{c_2}$ and $def_c^p(t) = def_{c_1}^p(t \downarrow^{con_c}_{con_{c_1}}) \times_p def_{c_2}^p(t \downarrow^{con_c}_{con_{c_2}})$.*

See [1] for the definition of the projection $t \downarrow^{W}_{W'}$ of a tuple t from a set W to a set W', and the definition of the projection $c \Downarrow I$ of a constraint $c = \langle def^p_c,\ con_c\rangle$ over set I of variables. A solution to an SCSP is a single constraint formed by the combination of all the original constraints. An *abstract solution* consists of the set of k-tuples of D whose c-semiring values are maximal w.r.t. $\leqslant_{S_p}$.

Definition 4. *Given an SCSP $P = \langle C,\ con\rangle$ over a constraint system CS, the solution of P is a constraint $Sol(P) = (\bigotimes C) = \langle def^p_c,\ con\rangle$ where $\bigotimes C = c_1 \otimes c_2 \otimes ... \otimes c_n$ with $C = \{c_1, ..., c_n\}$. The set $ASol(P) = \{\langle t, v\rangle \mid def^p_c(t) = v$ and there is no t' such that $v <_{S_p} def^p_c(t')\}$ is the abstract solution and $ASolV(P) = \{v \mid \langle t, v\rangle \in ASol(P)\}$ contains the maximal preference values.*

Definition 5. *[7] Let a good enough (abstract) solution for a SCSP P be such that some element in ASolV(P) is in the region $\hat{\beta}$ where $\hat{\beta} = \{\gamma \epsilon A : \beta \leqslant_{S_p} \gamma\}$.*

If $ASolV(P) \cap \hat{\beta} = \emptyset$ we want to find a relaxation P' of P, such that $ASolV(P') \cap \hat{\beta} \neq \emptyset$. P' should be as close to the original P as possible.

Definition 6. *A constraint $c_j = \langle def^p_j,\ con_j\rangle$ is called a c_i-weakened constraint of the constraint $c_i = \langle def^p_i,\ con_i\rangle$ iff the following hold: $con_i = con_j$; for all tuples t, $def^p_i(t) \leqslant_S def^p_j(t)$; and for every two tuples t_1 and t_2, if $def^p_i(t_1) <_{S_p} def^p_i(t_2)$, then $def^p_j(t_1) <_{S_p} def^p_j(t_2)$.*

Definition 7. *Given a constraint system $CS = \langle S_p,\ V,\ D\rangle$ and an SCSP $P = \langle C,\ con\rangle$, for each $c \in C$, let W_c be the set containing all c-weakened constraints, i.e. $W_c = \{c_j \mid c_j$ is a c-weakened constraint$\}$. Let $S_d = \langle A_d,\ +_d,\ \times_d,\ \mathbf{0}_d,\ \mathbf{1}_d\rangle$ be a c-semiring and $wdef^d_c : W_c \rightarrow A_d$ be any function such that the following hold: $wdef^d_c(c_j) = \mathbf{0}$ iff $c_j = c$; $\forall c_i, c_j \in W_c$, if for all tuples t $def^p_i(t) \leqslant_{S_p} def^p_j(t)$ then $wdef^d_c(c_i) \leqslant_{S_d} wdef^d_c(c_j)$; and if there exists one tuple t such that $def^p_i(t) <_{S_p} def^p_j(t)$ and for all tuples s we have $def^p_i(s) \leqslant_{S_p} def^p_j(s)$, then $wdef^d_c(c_i) <_{S_d} wdef_c d(c_j)$.*

Definition 8.
- *The c-weakened constraint c_i is closer to c than the c-weakened constraint c_j, iff $wdef^d_c(c_i) <_{S_d} wdef^d_c(c_j)$.*
- *The c-weakened constraint c_i is no closer to c than the c-weakened constraint c_j, iff $wdef^d_c(c_j) \leqslant_{S_d} wdef^d_c(c_i)$.*
- *The c-weakened constraints c_i and c_j are incomparable w.r.t. closeness to c iff $wdef^d_c(c_i) \nleqslant_{S_d} wdef^d_c(c_j)$ and $wdef^d_c(c_j) \nleqslant_{S_d} wdef^d_c(c_i)$.*

The function $wdef^d_c$ assigns a distance value from the set of the c-semiring S_d to each c-weakened constraint, and is restricted as follows. Let c_{ik} be a c_i-weakened constraint, and c_{jm} and c_{jn} be c_j-weakened constraints. If $wdef^d_{c_j}(c_{jm}) <_{S_d} wdef^d_{c_j}(c_{jn})$, then $wdef^d_{c_i}(c_{ik}) \times_d wdef^d_{c_j}(c_{jm}) <_{S_d} wdef^d_{c_i}(c_{ik}) \times_d wdef^d_{c_j}(c_{jn})$.

Definition 9. *A SCSP $P' = \langle C',\ con\rangle$ is a d-relaxation of the SCSP $P = \langle C,\ con\rangle$ where $S_d = \langle A_d,\ +_d,\ \times_d,\ \mathbf{0}_d,\ \mathbf{1}_d\rangle$, iff there is a bijection $f : C \rightarrow C'$ and $\forall c \in C, f(c)$ is a c-weakened constraint. Let $R(P) = \{P' \mid P'$ is a d-relaxation of $P\}$, and $R_{\hat{\beta}}(P) = \{P' \in R(P) \mid ASolV(P') \cap \hat{\beta} \neq \emptyset\}$.*

A d-relaxation $P' = \langle C', con\rangle$ of $P = \langle C, con\rangle$ is such that every c-weakened constraint $c' \in C'$ is the closest possible to the constraint $c \in C$ while the abstract solution of P' is still good enough (w.r.t. $\hat{\beta}$).

Definition 10. *Given a d-relaxation $P' = \langle C',\ con\rangle$ of a SCSP $P = \langle C,\ con\rangle$ such that $P' \in R_{\hat{\beta}}(P)$, let $d(P') = \times_{d\ c\in C}\ (wdef^d_c(f(c)))$ be the distance between P and P'. The set $MR_{\hat{\beta}}(P) = \{P' \in R_{\hat{\beta}}(P) \mid \nexists\ P'' \in R_{\hat{\beta}}(P)$ such that $d(P'') <_S d(P')\}$ contains the relaxations closest to P.*

3 A Combined Semiring

Definition 11. *Suppose $S_A = \langle A,\ \oplus_A,\ \otimes_A,\ \mathbf{0}_A,\ \mathbf{1}_A\rangle$ and $S_B = \langle B,\ \oplus_B,\ \otimes_B,\ \mathbf{0}_B,\ \mathbf{1}_B\rangle$ are two c-semirings. Let a Combined C-Semiring be $S_U = \langle U,\ \oplus_U,\ \otimes_U,\ \mathbf{0}_U,\ \mathbf{1}_U\rangle$ with $U = \{\langle\langle a_1, \ldots, a_k\rangle, b\rangle \mid a_i \in A$, and $b \in B\}$ for some fixed non-negative integer k. If we have $u_1, u_2 \in U$, with $u_1 = \langle\langle a_{1_1}, \ldots, a_{1_k}\rangle, b_1\rangle$ and $u_2 = \langle\langle a_{2_1}, \ldots, a_{2_k}\rangle, b_2\rangle$, then the following statements hold.*

- *$u_1 \otimes_U u_2 = \langle\langle a_{1_1} \otimes_A a_{2_1},\ \ldots, a_{1_k} \otimes_A a_{2_k}\rangle,\ b_1 \otimes_B b_2\rangle$.*
- *$u_1 \oplus_U u_2 = \langle\langle a_{1_1} \oplus_A a_{2_1},\ \ldots, a_{1_k} \oplus_A a_{2_k}\rangle,\ b_1 \oplus_B b_2\rangle$.*
- *$\mathbf{0}_U = \langle\langle a_1, \ldots, a_k\rangle, b\rangle$ such that every $a_i = \mathbf{0}_A$ and $b = \mathbf{0}_B$, and $\mathbf{1}_U = \langle\langle a_1, \ldots, a_k\rangle, b\rangle$ such that every $a_i = \mathbf{1}_A$ for $i = \{1, \ldots, k\}$, and $b = \mathbf{1}_B$.*
- *A pre-order $\leqslant_U$ over the set U is defined as $u_1 \leqslant_U u_2$ iff $b_1 \leqslant_B b_2$.*

Definition 12. *Let $P = \langle C, con\rangle$ be an SCSP over a constraint system $CS = \langle S_p, D, V\rangle$ and $P' = \langle C', con\rangle$ be a d-relaxation of P. A Combined Semiring Relaxation of P is a tuple $\langle P', g\rangle$ with $S_U = \langle U,\ +_U,\ \otimes_U,\ \mathbf{0}_U,\ \mathbf{1}_U\rangle$, where $g : C \times C' \rightarrow U$, i.e. for every $c = \langle def^p_c, con_c\rangle \in C$ and every c-weakened constraint $c_r \in C'$, $g(\langle c, c_r\rangle) = u_{cr}$ with $u_{cr} \in U$.*
Assume all tuples of values of D are strictly ordered. Let $u_{cr} = \langle Pref_{cr}, b_{cr}\rangle$, where b_{cr} is the distance value associated with the constraint c_r, and $Pref_{cr} = \langle a_{cr_1}, \ldots, a_{cr_k}\rangle$ where a_{cr_i}, for $i = \{1, \ldots, k\}$, are the preference values associated with the constraint $c_r \otimes_p c_{BEST}$. Let $c_{BEST} = \langle def^p_{c_{BEST}}, con\rangle$ be a dummy constraint with $def^p_{c_{BEST}}(t) = \mathbf{1}_p$ for every tuple t over the set of variables con.

Note that $def^p_{c_r \otimes_p c_{BEST}}(t) = def^p_{c_r}(t \downarrow^{con}_{con_c}) \otimes_p \mathbf{1}_p = def^p_{c_r}(t \downarrow^{con}_{con_c})(t)$. The coordinates in the set $Pref_{cr}$ are the preference values associated with the k tuples in the relaxed constraint (over the variables in the set con) while b_{cr} represents the distance between the relaxed problem P' and the original problem P.

Definition 13. *Given a Combined Semiring Relaxation $RP = \langle P', g\rangle$ of an SCSP $P = \langle C, con\rangle$ and a d-relaxation $P' = \langle C', con\rangle$, the solution of RP is a constraint defined as $RSol(RP) = (\bigotimes C')$ with $g(\bigotimes C, \bigotimes C') = u_{CR}$. Suppose $u_{CR} = \langle Pref_{CR}, b_{CR}\rangle$, and $Pref_{CR} = \langle a_{CR_1}, \ldots, a_{CR_k}\rangle$. Then the abstract solution of RP is the set $RASol(RP) = \{\langle t, a\rangle \mid a \in Pref_{CR}, t$ is the tuple with which a is associated, and there is no $\langle t', a'\rangle$ such that $a <_{S_p} a'\}$. Let $ASolV(RP) = \{a \mid \langle t, a\rangle \in RASol(RP)\}$.*

Let $Rel(P) = \{RP \mid RP$ is a Combined Semiring Relaxation of an SCSP $P = \langle C, con \rangle\}$. Now we define a set containing the best Combined Semiring Relaxations with solutions that are good enough.

Definition 14. *Suppose for every* $RP = \langle P', g \rangle \in Rel(P)$ *with* $P' = \langle C', con \rangle$, *we have* $g(\bigotimes C, \bigotimes C') = u_{CR} = \langle Pref_{CR}, b_{CR} \rangle$, *and* $Pref_{CR} = \langle a_{CR_1}, \ldots, a_{CR_k} \rangle$. *Let* $Rel_{\hat{\alpha}}(P) = \{RP \in Rel(P) \mid RASolV(RP) \cap \hat{\alpha} \neq \emptyset$ *and there is no relaxation* $PR' \in Rel(P)$ *such that* $b_{CR} <_{S_d} b_{CR'}\}$.

4 Conclusion and Future Work

If the preference value associated with the abstract solution of an SCSP is not regarded as good enough, a suitable relaxation of the SCSP that has a good enough solution is found by adjusting the preferences associated with the tuples of some of the constraints (i.e. c-semiring values of the first semiring) of the original SCSP. In other words, the constraints of the original problem are relaxed until the resulting problem has a satisfactory solution. Distance values (i.e. c-semiring values from a second semiring) are associated with each relaxed constraint so that different relaxations of a problem can be compared in terms of their distance to the original problem. In this paper we show how to combine these two semirings into a single semiring.

The combined semiring allows us to rely on existing techniques for solving SCSPs. At this stage we simply have a technical result and need to investigate computational aspects of this process. We aim to develop techniques to calculate solutions to a maximal Combined Relaxation of SCSP efficiently.

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