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Refined Epistemic Entrenchment

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Abstract. Epistemic entrenchment, as presented by Gärdenfors and Makinson (1988) and Gärdenfors (1988), is a formalisation of the intuition that, when forced to choose between two beliefs, an agent will give up the less entrenched one. While their formalisation satisfactorily captures the intuitive notion of the entrenchment of beliefs in a number of aspects, the requirement that all wffs be comparable has drawn criticism from various quarters. We define a set of refined versions of their entrenchment orderings that are not subject to the same criticism, and investigate the relationship between the refined entrenched orderings, the entrenchment orderings of Gärdenfors and Makinson, and AGM theory contraction (Alchourrón et al., 1985). To conclude, we compare refined entrenchment with two related approaches to epistemic entrenchment.

Key words: belief contraction, belief revision, epistemic entrenchment, power orders, theory change

1. Introduction

Epistemic entrenchment, as presented by Gärdenfors and Makinson (1988) and Gärdenfors (1988), formalises the intuition that, when forced to choose between two beliefs, an agent will give up the less entrenched one. Their entrenchment orderings, which we refer to as the EE-orderings, satisfactorily capture the intuitive notion of the entrenchment of beliefs in a number of aspects, but also suffer from some drawbacks. Foremost among the drawbacks is the insistence that all wffs be comparable. In this paper we define a set of refined versions of the EE-orderings that are not subject to the same criticism. The definition of the refined entrenchment

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orderings is semantic. It is based on a well-known result regarding the construction of the EE-orderings in terms of faithful total preorders on the interpretations of the underlying logic language. We also provide a characterisation via a set of postulates.

We investigate the relationship between the EE-orderings, the refined entrenchment orderings and the AGM approach to theory contraction (Alchourrón et al., 1985). In particular, we show that the refined entrenchment orderings and the EE-orderings are interdefinable, and that the same goes for refined entrenchment and AGM theory contraction. We conclude with a comparison of refined entrenchment and related approaches to epistemic entrenchment.

1.1. PRELIMINARIES

For the rest of this paper L denotes any logic language, closed under the usual propositional connectives, and containing the symbols $\top$ and $\bot$. We assume L to have a two-valued model-theoretic semantics defining truth and falsity. The set of interpretations of L is denoted by U. We use $\Vdash$ for the relation from U to L denoting satisfaction and we assume that $\Vdash$ behaves classically with respect to the propositional connectives. It thus follows that for every $u \in U$ and every $\alpha \in L$, $u \Vdash \alpha$ or $u \Vdash \neg\alpha$. We use $\top$ and $\bot$ as canonical representatives for the logically valid and logically invalid wffs respectively. For concreteness the reader may think of the logic under consideration as a (possibly infinitely generated) propositional logic. For every $X \subseteq L$, $M(X) = \{x \in U \mid x \Vdash \alpha$ for every $\alpha \in X\}$ is the set of *models* of X, and for $\alpha \in L$ we write $M(\alpha)$ instead of $M(\{\alpha\})$. Entailment (from $\wp L$ to L) for L is defined as follows: $X \vDash \beta$ iff $M(X) \subseteq M(\beta)$, and for $\alpha, \beta \in L$ we write $\alpha \vDash \beta$ instead of $\{\alpha\} \vDash \beta$. We also require of $\vDash$ to satisfy *compactness*. That is, for every $X \subseteq L$ and every $\alpha \in L$, $X \vDash \alpha$ iff $X_F \vDash \alpha$ for some finite subset X_F of X. By $\alpha \equiv \beta$ we understand that α and β are logically equivalent, i.e., $\alpha \vDash \beta$ and $\beta \vDash \alpha$. Closure under entailment is denoted by Cn. A *theory* or a *belief set* is a set $K \subseteq L$ closed under entailment. A set $X \subseteq L$ is *satisfiable* iff $M(X) \neq \emptyset$, iff $Cn(X) \neq L$. For every satisfiable subset X of L, $\alpha \in L$ is X*-valid* iff $X \vDash \alpha$, α is X*-undecided* iff $X \nvDash \alpha$ and $X \nvDash \neg\alpha$, and α is X*-invalid* iff $X \vDash \neg\alpha$. For an unsatisfiable subset X of L, all the wffs of L are X-valid, while none are X-undecided or X-invalid. For every $V \subseteq U$, the *theory determined by* V is $Th(V) = \{\alpha \in L \mid x \Vdash \alpha$ for every $x \in V\}$, and for $x \in U$ we write $Th(x)$ instead of $Th(\{x\})$. For a belief set K, the *expansion* of K by a wff $\alpha \in L$ is defined as $K + \alpha = Cn(K \cup \{\alpha\})$. Given a preorder $\sqsubseteq$ (i.e., a reflexive transitive relation) on any set W, we write $x \sqsubset y$ iff $x \sqsubseteq y$ and $y \not\sqsubseteq x$, $x \equiv_{\sqsubseteq} y$ iff $x \sqsubseteq y$ and $y \sqsubseteq x$, $x \parallel_{\sqsubseteq} y$ iff $x \not\sqsubseteq y$ and $y \not\sqsubseteq x$, and we let $[x]_{\sqsubseteq} = \{y \mid x \equiv_{\sqsubseteq} y\}$.

In the original AGM papers, the underlying logic language L is associated with a consequence relation $\vdash$ and the consequence operator Cn generated by $\vdash$ is assumed to be a Tarskian closure operation (for $X, Y \subseteq L$, $X \subseteq Cn(X)$, $Cn(Cn(X)) \subseteq Cn(X)$ and $Cn(X) \subseteq Cn(Y)$ if $X \subseteq Y$), to be supraclassical (to include all truth-functional tautologies in every $Cn(X)$, and to satisfy Modus Ponens), to satisfy the deduction theorem (for $X \subseteq L$ and $\alpha, \beta \in L$, $\beta \in Cn(X \cup \{\alpha\})$ iff $\alpha \rightarrow \beta \in Cn(X)$), and to be compact. It is easy to show that the entailment relations we consider are precisely the consequence relations allowed by AGM. In our case they are just described semantically. To see why, note firstly that every entailment relation $\vDash$ we consider clearly satisfies the four properties outlined above. And conversely, from every consequence relation $\vdash$ we can construct an appropriate semantics for L that will satisfy all the requirements set out above. Simply take U, the set of interpretations of L, to be the set of maximally consistent subsets of L. That is, let $U = \{X \in L \mid X \nvdash \bot$ and $\forall Y \subseteq L$ s.t. $X \subset Y, Y \vdash \bot\}$. The satisfaction relation $\Vdash$ is then defined as follows: For every $\alpha \in L$ and $X \in U$, $X \Vdash \alpha$ iff $\alpha \in X$. It is readily verified that the semantic entailment relation $\vDash$ obtained from $\Vdash$ behaves exactly like $\vdash$.

2. AGM Theory Change

AGM theory contraction and revision can be described in terms of sets of postulates. Since all the AGM postulates deal with fixed belief sets, we shall assume a fixed belief set K and define contraction and revision functions as functions from L to the set of belief sets. The eight postulates for AGM revision are given below.

$(K*1)$ $K*\phi = Cn(K*\phi)$.

$(K*2)$ $\phi \in K*\phi$.

$(K*3)$ $K*\phi \subseteq K+\phi$.

$(K*4)$ If $\neg\phi \notin K$, then $K+\phi \subseteq K*\phi$.

$(K*5)$ $K*\phi = L$ iff $\vDash \neg\phi$.

$(K*6)$ If $\phi \equiv \psi$ then $K*\phi = K*\psi$.

$(K*7)$ $K*(\phi \wedge \psi) \subseteq (K*\phi)+\psi$.

$(K*8)$ If $\neg\psi \notin K*\phi$, then $(K*\phi)+\psi \subseteq K*(\phi \wedge \psi)$.

The AGM contraction postulates follow a similar pattern.

$(K-1)$ $K-\phi = Cn(K-\phi)$.

$(K-2)$ $K-\phi \subseteq K$.

$(K-3)$ If $\phi \notin K$ then $K-\phi = K$.

$(K-4)$ If $\nvDash \phi$ then $\phi \notin K-\phi$.

$(K-5)$ If $\phi \in K$ then $(K-\phi)+\{\phi\} = K$.

$(K-6)$ If $\phi \equiv \psi$ then $K - \phi = K - \psi$.

$(K-7)$ $(K - \phi) \cap (K - \psi) \subseteq K - (\phi \wedge \psi)$.

$(K-8)$ If $\psi \notin K - (\phi \wedge \psi)$ then $K - (\phi \wedge \psi) \subseteq K - \psi$.

The following two identities can be used to define revision and contraction in terms of one another.

(Harper Identity) $K - \phi = K \cap (K * \neg\phi)$.

(Levi Identity) $K * \phi = (K - \neg\phi) + \phi$.

It is well known that the application of the Harper Identity to a revision function satisfying the eight AGM revision postulates yields a contraction function that satisfies the eight AGM contraction postulates. Similarly, the application of the Levi Identity to a contraction function $-$, satisfying the eight AGM contraction postulates, yields a revision function $*$ that satisfies the eight AGM revision postulates. Moreover, the contraction function $\div$ obtained from $*$ via another application of the Harper Identity is identical to $-$ (see, for example, Gärdenfors, 1988).

From results in Grove (1988), Katsuno and Mendelzon (1991) and Boutilier (1994), AGM theory change can be characterised by a set of preorders on U.

DEFINITION 1. Let $\preceq$ be any preorder on U.

1. $x \in V \subseteq U$ is $\preceq$-*minimal* in V iff for every $y \in V$, $y \not\prec x$.
2. $V \subseteq U$ is $\preceq$-*smooth* iff for every $y \in V$ there is an x that is $\preceq$-minimal in V such that $x \preceq y$.
3. $\preceq$ is *smooth* iff $M(\phi)$ is $\preceq$-smooth for every ϕ. We denote the set of $\preceq$-minimal elements of $M(\phi)$ by $\text{Min}_{\preceq}(\phi)$.
4. A preorder $\preceq$ on U is *faithful* (with respect to K) iff $\preceq$ is smooth, $x \prec y$ for every $x \in M(K)$ and $y \notin M(K)$, and $x \not\prec y$ for every $x, y \in M(K)$.

The idea is to consider preorders in which the models of K, being the minimal, or "best" interpretations, are strictly below all other interpretations.

DEFINITION 2.

1. A revision function $*$ is *constructed from* a faithful preorder $\preceq$ iff $K * \phi = Th(\text{Min}_{\preceq}(\phi))$.
2. A contraction function $-$ is *constructed from* a faithful preorder $\preceq$ iff $K - \phi = Th(M(K) \cup \text{Min}_{\preceq}(\neg\phi))$.

A reinterpretation of Grove's "systems of spheres" in (1988) as faithful *total* preorders yields a semantic characterisation of AGM revision, in the same spirit as that of Katsuno and Mendelzon (1991), and Boutilier (1994).* Combined with the results about the Levi and Harper identities mentioned above, it provides a semantic characterisation of AGM contraction as well. We state these results without proof in the theorem below, and use it throughout the rest of this paper without explicit references to it.

THEOREM 1.
1. *A revision function satisfies* $(K * 1)$ *to* $(K * 8)$ *iff it is constructed from some faithful total preorder.*
2. *A contraction function satisfies* $(K - 1)$ *to* $(K - 8)$ *iff it is constructed from some faithful total preorder.*

2.1. Epistemic Entrenchment

An *EE-ordering* $\sqsubseteq_{\mathrm{EE}}$ (with respect to K) is an ordering on all wffs of L – the wffs higher up in the ordering being more entrenched – where $\sqsubseteq_{\mathrm{EE}}$ is subject to the following set of postulates:

$(EE1)$ $\sqsubseteq_{\mathrm{EE}}$ is transitive.

$(EE2)$ If $\phi \vDash \psi$ then $\phi \sqsubseteq_{\mathrm{EE}} \psi$.

$(EE3)$ $\phi \sqsubseteq_{\mathrm{EE}} \phi \wedge \psi$ or $\psi \sqsubseteq_{\mathrm{EE}} \phi \wedge \psi$.

$(EE4)$ If $K \neq L$ then $\phi \notin K$ iff for all ψ, $\phi \sqsubseteq_{\mathrm{EE}} \psi$.

$(EE5)$ If $\phi \sqsubseteq_{\mathrm{EE}} \psi$ for all ϕ, then $\vDash \psi$.

It is easy to show that the EE-orderings are total preorders. The following theorem shows how to construct contraction functions from EE-orderings.

THEOREM 2 (Gärdenfors and Makinson, 1988).
1. *If* $\sqsubseteq_{\mathrm{EE}}$ *is an EE-ordering, then the contraction function* $-$ *defined below satisfies all eight of the AGM contraction postulates:*

$$\psi \in K - \phi \text{ iff } \psi \in K \text{ and } (\phi \sqsubset_{\mathrm{EE}} (\phi \vee \psi) \text{ or } \vDash \phi).$$

2. *If* $-$ *is a contraction function satisfying the eight AGM contraction postulates, then the relation* $\sqsubseteq$ *defined below is an EE-ordering:*

$$\phi \sqsubseteq \psi \text{ iff } \vDash \psi \text{ or } \phi \notin K - (\phi \wedge \psi).$$

* The logic languages we consider may have a slightly more general semantics than those considered by Grove. In particular, it may contain distinct interpretations which satisfy exactly the same set of wffs.

2.2. Power Orders

A particularly interesting result about the EE-orderings is that they can be constructed from the faithful total preorders. The idea is to lift a faithful total preorder in a sensible way to a *power order* (an ordering on *sets* of interpretations). Because every wff is associated with a particular set of interpretations – its set of models – we can view the ordering on sets of interpretations as an ordering on wffs. For reasons that will become clear, we define the appropriate power-construction so that it can be applied to all the faithful preorders (and not just the faithful *total* preorders).

DEFINITION 3. Let $\preceq$ be any faithful preorder. The *power order* $\sqsubseteq_{EP}$ on L, induced by $\preceq$, is defined as: $\phi \sqsubseteq_{EP} \psi$ iff for every $y \in M(\neg\psi)$ there is an $x \in M(\neg\phi)$ such that $x \preceq y$.

From results in Grove (1988), Gärdenfors (1988), Boutilier (1992, 1994), it follows that this construction applied to the faithful total preorders induce precisely the EE-orderings.

THEOREM 3. *A relation $\sqsubseteq$ on L is an EE-ordering iff it is the power order induced by some faithful total preorder.*

This brings us to a few useful technical results, which we state without proof. We shall make use of these results without always referring to them explicitly.

LEMMA 1. *Let $\preceq$ be a faithful preorder (not necessarily total), and let $\sqsubseteq$ be the power order induced by $\preceq$.*

1. *$\phi \sqsubseteq \psi$ iff for every $y \in \mathrm{Min}_{\preceq}(\neg\psi)$ there is an $x \in \mathrm{Min}_{\preceq}(\neg\phi)$ such that $x \preceq y$.*
2. *$\phi \not\sqsubseteq \psi$ iff there is a $y \in \mathrm{Min}_{\preceq}(\neg\psi)$ such that $x \in M(\phi)$ for every $x \preceq y$.*

When applied to the faithful total preorders and their induced EE-orderings, Lemma 1 shows that two wffs are equally entrenched if and only if the minimal models of their negations are on the same level, and for any two wffs ϕ and ψ that are both not logically valid, ϕ is strictly more entrenched than ψ if and only if the minimal models of $\neg\phi$ are strictly below the minimal models of $\neg\psi$.

COROLLARY 1. *Let $\preceq$ be a faithful total preorder, and let $\sqsubseteq_{\mathrm{EE}}$ be the EE-ordering induced by $\preceq$.*

1. *$\phi \equiv_{\sqsubseteq_{\mathrm{EE}}} \psi$ iff for every $y \in \mathrm{Min}_{\preceq}(\neg\psi)$ and every $x \in \mathrm{Min}_{\preceq}(\neg\phi)$, $x \equiv_{\preceq} y$.*
2. *If $\not\models \phi$ and $\not\models \psi$ then $\phi \sqsubset_{\mathrm{RE}} \psi$ iff $x \prec y$ for every $y \in \mathrm{Min}_{\preceq}(\neg\psi)$ and every $x \in \mathrm{Min}_{\preceq}(\neg\phi)$.*

The next proposition provides an expected result about the connection between the EE-orderings and the faithful total preorders.

PROPOSITION 1 . *The contraction function constructed from a faithful total preorder* $\preceq$, *and the one obtained from the EE-ordering induced by* $\preceq$ *(by the construction in Theorem 2), are identical.*

Proof. We only consider the case where $\phi, \psi \in K, \nvDash \phi$, and $\nvDash \psi$, the remaining cases are trivial. It suffices to show that $\mathrm{Min}_{\preceq}(\neg\phi) \subseteq M(\psi)$ iff $\phi \sqsubset_{\mathrm{EE}} \phi \vee \psi$. Now, $\mathrm{Min}_{\preceq}(\neg\phi) \subseteq M(\psi)$ iff $y \in M(\psi)$ for every $y \in \mathrm{Min}_{\preceq}(\neg\phi)$, iff there is a $y \in M(\neg\phi)$ such that $x \in M(\phi \vee \psi)$ for every $x \preceq y$, iff $\phi \vee \psi \not\sqsubseteq_{\mathrm{EE}} \phi$, iff $\phi \sqsubset_{\mathrm{EE}} \phi \vee \psi$. □

3. Refined Entrenchment

The EE-orderings of Gärdenfors provide a satisfactory formalisation of the intuitive notion of the entrenchment of beliefs in almost all aspects. However, the insistence that every wff be comparable to every other one (due to the fact that the EE-orderings are total preorders) has drawn criticism from various quarters (Lindström and Rabinowicz, 1991; Rott, 1992). The main purpose of this paper is to present and investigate refined versions of the EE-orderings that are not subject to the same criticism. These orderings are obtained by applying the power-construction of Definition 3, not to the faithful total preorders, but to a closely related set of faithful preorders.

DEFINITION 4. A weak partial order $\leq$ on a set V is called *modular* iff for every $x, y, z \in V$, if $x \nleq y$, $y \nleq x$, and $z \leq x$, then $z \leq y$.

The modular weak partial orders are the reflexive versions of the modular partial orders of Ginsberg (1986) and Lehmann and Magidor (1992). Intuitively, a modular weak partial order ensures that the elements of V are arranged in levels, with incomparable elements being regarded as on the same level. Using this intuition, it is clear that there is a natural connection between the total preorders and the modular weak partial orders.

DEFINITION 5.

1. Given a total preorder $\preceq$ on a set V, the relation

$$\leq \;=\; \preceq \;\setminus\; \{(x, y) \in V \times V \mid x \neq y, x \preceq y, \text{ and } y \preceq x\}$$

is the modular weak partial order *associated with* $\preceq$.

2. Given a modular weak partial order $\leq$ on a set V, the relation

$$\preceq \;=\; \leq \;\cup\; \{(x, y) \in V \times V \mid x \nleq y \text{ and } y \nleq x\}$$

is the total preorder *associated with* $\leq$.

The next proposition shows that the shift from the faithful total preorders to the faithful modular weak partial orders is an inessential technical modification if we are only interested in minimality.

PROPOSITION 2. *The contraction and revision functions constructed from a faithful total preorder can also be constructed from its associated faithful modular weak partial order.*

Proof. The proof follows from the fact that for every faithful total preorder $\preceq$ and its associated modular weak partial order $\leq$, $\mathrm{Min}_{\preceq}(\phi) = \mathrm{Min}_{\leq}(\phi)$ for every $\phi \in L$. $\square$

As we shall see, however, the power-construction of Definition 3 is more sensitive to such a shift.

DEFINITION 6. An *RE-ordering* (refined entrenchment ordering) $\sqsubseteq_{\mathrm{RE}}$ is a power order induced by a faithful modular weak partial order $\leq$ (i.e., $\phi \sqsubseteq_{\mathrm{RE}} \psi$ iff for every $y \in M(\neg\psi)$ there is an $x \in M(\neg\phi)$ such that $x \leq y$). For a faithful total preorder $\preceq$ and its associated faithful modular weak partial order $\leq$, the EE-ordering and RE-ordering induced by $\preceq$ and $\leq$ are said to be *associated with one another*.

The following proposition provides a preliminary list of properties of the RE-orderings.

PROPOSITION 3. *An RE-ordering $\sqsubseteq_{\mathrm{RE}}$ has the following properties.*

1. *$\sqsubseteq_{\mathrm{RE}}$ is a preorder (that need not be total).*
2. *If $\phi \sqsubseteq_{\mathrm{RE}} \psi$ then $\phi \sqsubseteq_{\mathrm{EE}} \psi$, where $\sqsubseteq_{\mathrm{EE}}$ is the EE-ordering associated with $\sqsubseteq_{\mathrm{RE}}$.*
3. *If $\phi \vDash \psi$ then $\phi \sqsubseteq_{\mathrm{RE}} \psi$.*
4. *$\phi \sqsubseteq_{\mathrm{RE}} \psi$ for all ϕ iff $\vDash \psi$.*
5. *If $\phi \equiv \psi$ then $\phi \sqsubseteq_{\mathrm{RE}} \chi$ iff $\psi \sqsubseteq_{\mathrm{RE}} \gamma$, and $\chi \sqsubseteq_{\mathrm{RE}} \phi$ iff $\chi \sqsubseteq_{\mathrm{RE}} \psi$.*
6. *If K is satisfiable then $\{\phi \mid \neg\phi \in K\} = [\bot]_{\sqsubseteq_{\mathrm{RE}}}$.*
7. *If $\phi \notin K$ and $\psi \in K$ then $\phi \sqsubset_{\mathrm{RE}} \psi$.*
8. *If $\neg\psi \in K$ and $\neg\chi \notin K$ then $\psi \sqsubset_{\mathrm{RE}} \chi$.*
9. *If $\phi \notin K$ then $K \cup \{\phi\} \vDash \psi$ iff $\phi \sqsubseteq_{\mathrm{RE}} \psi$.*
10. *If $\phi \equiv_{\sqsubseteq_{\mathrm{RE}}} \psi$ then $\phi \wedge \psi, \phi \vee \psi \in [\phi]_{\sqsubseteq_{\mathrm{RE}}} = [\psi]_{\sqsubseteq_{\mathrm{RE}}}$.*
11. *$\phi \sqsubseteq_{\mathrm{RE}} \phi \wedge \psi$, or $\psi \sqsubseteq_{\mathrm{RE}} \phi \wedge \psi$, or both $\phi \to \psi \not\sqsubseteq_{\mathrm{RE}} \phi$ and $\psi \to \phi \not\sqsubseteq_{\mathrm{RE}} \psi$.*

Proof. Let $\leq$ be the faithful modular weak partial order that induces $\sqsubseteq_{\mathrm{RE}}$. The reflexivity and transitivity of $\sqsubseteq_{\mathrm{RE}}$ are trivial. To show that $\sqsubseteq_{\mathrm{RE}}$ need not be a total preorder, consider the example of a propositional language generated by two atoms, p and q. Now let $K = Cn(p)$ and consider the faithful modular weak partial order

$$\{(x, x) \mid x \in U\} \cup \{(x, y) \mid x \in M(K) \text{ and } y \notin M(K)\}.$$

It is easily verified that $q \not\sqsubseteq_{\mathrm{RE}} \neg q$ and $\neg q \not\sqsubseteq_{\mathrm{RE}} q$. (2) follows from the fact that if $x \leq y$ then $x \preceq y$ where $\preceq$ is the faithful total preorder associated with $\leq$. (3) is trivial. For (4), suppose that $\phi \sqsubseteq_{\mathrm{RE}} \psi$ for all ϕ. So in particular $\top \sqsubseteq_{\mathrm{RE}} \psi$, which can only be if $M(\neg\psi) = \emptyset$. Therefore, $\vDash \psi$. Conversely, if $\vDash \psi$ then $M(\neg\psi) = \emptyset$, from which it follows vacuously that $\phi \sqsubseteq_{\mathrm{RE}} \psi$ for all ϕ. (5) is trivial. For (6), suppose that K is satisfiable and pick any ϕ such that $\neg\phi \in K$. $\bot \sqsubseteq_{\mathrm{RE}} \phi$ follows from $\bot \vDash \phi$, and $\phi \sqsubseteq_{\mathrm{RE}} \bot$ from the faithfulness of $\leq$. Conversely, pick any $\phi \in [\bot]_{\sqsubseteq_{\mathrm{RE}}}$ and assume that $\neg\phi \notin K$. Then there is at least one model x of K that satisfies ϕ, and thus $\phi \not\sqsubseteq_{\mathrm{RE}} \bot$, contradicting the supposition that $\phi \in [\bot]_{\sqsubseteq_{\mathrm{RE}}}$. For (7), suppose $\phi \notin K$ and $\psi \in K$. So $M(K) \cap M(\neg\psi) = \emptyset$, and since K has a model that satisfies $\neg\phi$, it follows from faithfulness that for every $y \in M(\neg\psi)$ there is an $x \in M(\neg\phi)$ such that $x \leq y$, i.e., $\phi \sqsubseteq_{\mathrm{RE}} \psi$. On the other hand, since K has a model y that satisfies $\neg\phi$, and since all models of K satisfy ψ, it follows from faithfulness that $x \in M(\psi)$ for every $x \leq y$, and thus $\psi \not\sqsubseteq_{\mathrm{RE}} \phi$. For (8), suppose that $\neg\psi \in K$ and $\neg\chi \notin K$. $\psi \sqsubseteq_{\mathrm{RE}} \chi$ follows from faithfulness. And since $M(K) \cap M(\chi) \neq \emptyset$, it follows from faithfulness that there is a $y \in M(\chi) \cap M(K) \subseteq M(\neg\gamma)$, such that $x \in M(\chi)$ for every $x \leq y$, i.e., $\chi \not\sqsubseteq_{\mathrm{RE}} \gamma$. For the proof of (9), let $\phi \notin K$ and suppose that $K \cup \{\phi\} \vDash \psi$. Now pick any $y \in M(\neg\psi)$. If $y \notin M(K)$ then, because $M(K) \cap M(\neg\phi) \neq \emptyset$, there is an $x \in M(K) \cap M(\neg\phi)$ such that $x \leq y$. And if $y \in M(K)$ then, because $M(K) \cap M(\phi) \subseteq M(\psi)$, $y \notin M(\neg\phi)$, and there is thus an $x \in M(\neg\phi)$ such that $x \leq y$. So $\phi \sqsubseteq_{\mathrm{RE}} \psi$. Conversely, suppose that $K \cup \{\phi\} \nvDash \psi$. So there is a $y \in M(K) \cap M(\phi)$ such that $y \in M(\neg\psi)$. That is, $y \in M(\neg\psi)$ and for every $x \leq y$, $x \in M(\phi)$, which means that $\phi \not\sqsubseteq_{\mathrm{RE}} \psi$. For the proof of (10), suppose that $\phi \equiv_{\sqsubseteq_{\mathrm{RE}}} \psi$. By part (3), $\phi \wedge \psi \sqsubseteq_{\mathrm{RE}} \phi$ and $\phi \sqsubseteq_{\mathrm{RE}} \phi \vee \psi$. Now assume that $\phi \vee \psi \not\sqsubseteq_{\mathrm{RE}} \phi$. Then there is a $y \in \mathrm{Min}_{\leq}(\neg\phi)$ such $x \in M(\phi \vee \psi)$ for every $x \leq y$. Therefore, $y \in M(\neg\phi) \cap M(\psi)$. But because $\psi \sqsubseteq_{\mathrm{RE}} \phi$, there is a $z \in \mathrm{Min}_{\leq}(\neg\psi)$ such that $z < y$ which, together with the minimality of y in $M(\phi)$, contradicts $\phi \sqsubseteq_{\mathrm{RE}} \psi$. To show that $\phi \sqsubseteq_{\mathrm{RE}} \phi \wedge \psi$, pick a $y \in M(\neg\phi \vee \neg\psi)$. If $y \in M(\neg\psi)$ then $\phi \sqsubseteq_{\mathrm{RE}} \psi$ guarantees that there is an $x \in M(\neg\phi)$ such that $x \leq y$. And the case where $y \in M(\neg\phi)$ is trivial. For the proof of (11), suppose that $\phi \not\sqsubseteq_{\mathrm{RE}} \phi \wedge \psi$ and $\psi \not\sqsubseteq_{\mathrm{RE}} \phi \wedge \psi$. Then there is a $y \in \mathrm{Min}_{\leq}(\neg\phi \vee \neg\psi)$ such that $x \in M(\phi)$ for every $x \leq y$, and there is a $v \in \mathrm{Min}_{\leq}(\neg\phi \vee \neg\psi)$ such that $u \in M(\psi)$ for every $u \leq v$. So $y \in M(\phi) \cap M(\neg\psi)$ and $x \in M(\phi) \cap M(\psi)$ for every $x < y$. Similarly, $v \in M(\neg\phi) \cap M(\psi)$ and $u \in M(\phi) \cap M(\psi)$ for every $u < v$. Since $\leq$ is a modular weak partial order, it therefore has to be the case that $y \parallel_{\leq} v$. So $y \in M(\neg\psi)$ and $x \in M(\psi \rightarrow \phi)$ for every $x \leq y$. That is, $\psi \rightarrow \phi \not\sqsubseteq_{\mathrm{RE}} \psi$. And similarly for v, $\phi \rightarrow \psi \not\sqsubseteq_{\mathrm{RE}} \phi$. □

An inspection of the properties set out in Proposition 3 reveals something of the structure of the RE-orderings. They are refined versions of the EE-orderings that need not be total. Furthermore, every RE-ordering partitions the set of wffs into four disjoint sets. The logically valid wffs are all equally entrenched and strictly more entrenched than all other wffs. Next comes the remaining wffs in K. While strictly more entrenched than the wffs not in K, they need not all be comparable. The third partition consists of the K-undecided wffs, which are all strictly less entrenched than the wffs in K and more entrenched than the K-invalid wffs. (If K is unsatisfiable, there are no K-undecided wffs or K-invalid wffs.) So the RE-orderings are able to distinguish between wffs not in K. In fact, the part of an RE-ordering restricted to the wffs that are not in K corresponds to classical entailment relative to K. This certainly has more intuitive appeal than regarding all the wffs that are not in K as equally entrenched, as the EE-orderings do. For example, it makes much more sense to regard a wff that is K-invalid as less entrenched than a wff that is merely K-undecided, than to regard them both as equally entrenched.

The last two parts of Proposition 3 are worth singling out. Note that part (10) does not hold for the EE-orderings. An interesting example is the case of a wff ϕ and its negation. In an EE-ordering $\sqsubseteq_{\text{EE}}$, it is perfectly acceptable to have $\neg\phi \equiv_{\sqsubseteq_{\text{EE}}} \phi$ as long as ϕ is not logically valid or logically invalid. However, if this were the case in an RE-ordering $\sqsubseteq_{\text{RE}}$, part (10) of Proposition 3 would require that $\phi \vee \neg\phi \in [\phi]_{\sqsubseteq_{\text{RE}}}$, thus contradicting part (4) of the same proposition. Part (11) bears a vague resemblance to the postulate ($EE3$), and will be used in our characterisation of the RE-orderings in terms of postulates. In fact, so will the properties contained in the lemma below.

LEMMA 2. *Let* $\sqsubseteq_{\text{RE}}$ *be an RE-ordering.*

1. *If* $\phi \to \chi \sqsubseteq_{\text{RE}} \phi$ *then* $\phi \to \psi \sqsubseteq_{\text{RE}} \phi$ *or* $\psi \to \chi \sqsubseteq_{\text{RE}} \psi$.
2. *If* $\phi \to \chi \sqsubseteq_{\text{RE}} \phi$ *then* $\phi \not\sqsubseteq_{\text{RE}} \psi$ *or* $\psi \to \chi \sqsubseteq_{\text{RE}} \psi$.
3. *If* $\phi \to \chi \sqsubseteq_{\text{RE}} \phi$ *then* $\psi \not\sqsubseteq_{\text{RE}} \chi$ *or* $\phi \to \psi \sqsubseteq_{\text{RE}} \phi$.

Proof. Let $\leq$ be the faithful modular weak partial order that induces $\sqsubseteq_{\text{RE}}$.

1. Suppose that $\phi \to \psi \not\sqsubseteq_{\text{RE}} \phi$ and $\psi \to \chi \not\sqsubseteq_{\text{RE}} \psi$. By $\phi \to \psi \not\sqsubseteq_{\text{RE}} \phi$ there is a $y \in \text{Min}_{\leq}(\neg\phi)$ such $x \in M(\phi \to \psi)$ for every $x \leq y$. And by the minimality of y in $M(\neg\phi)$, $x \in M(\phi) \cap M(\psi)$ for every $x < y$. Similarly, $\psi \to \chi \not\sqsubseteq_{\text{RE}} \psi$ implies that there is a $v \in \text{Min}_{\leq}(\neg\psi)$ such that $u \in M(\psi) \cap M(\chi)$ for every $u < v$. Since $\leq$ is a modular weak partial order, it has to be the case that $v \not< y$. And this means that $z \in M(\phi) \cap M(\chi)$ for every $z < y$. So $y \in M(\neg\phi)$ and $x \in M(\phi \to \chi)$ for every $x \leq y$. That is, $\phi \to \chi \not\sqsubseteq_{\text{RE}} \phi$.
2. Suppose that $\phi \sqsubseteq_{\text{RE}} \psi$ and $\psi \to \chi \not\sqsubseteq_{\text{RE}} \psi$. As for part (1), $\psi \to \chi \not\sqsubseteq_{\text{RE}} \psi$ means there is a $v \in \text{Min}_{\leq}(\neg\psi)$ such that $u \in M(\psi) \cap M(\chi)$ for every $u < v$. So by $\phi \sqsubseteq_{\text{RE}} \psi$ there is a $w \leq v$ such that $w \in \text{Min}_{\leq}(\neg\phi)$. And since $w \leq v$, it follows that $u \in M(\phi) \cap M(\chi)$ for every $u < w$. So $u \in M(\phi \to \chi)$ for every $u \leq w$. That is, $\phi \to \chi \not\sqsubseteq_{\text{RE}} \phi$.
3. Suppose that $\psi \sqsubseteq_{\text{RE}} \chi$ and $\phi \to \psi \not\sqsubseteq_{\text{RE}} \phi$. As in part (1), $\phi \to \psi \not\sqsubseteq_{\text{RE}} \phi$ means there is a $y \in \text{Min}_{\leq}(\neg\phi)$ such that $x \in M(\phi) \cap M(\psi)$ for every $x < y$. So by $\psi \sqsubseteq_{\text{RE}} \chi$, $z \not< y$ for every $z \in M(\neg\chi)$. And therefore $x \in M(\chi)$ for

every $x < y$. So $y \in M(\neg\phi)$ and $x \in M(\phi \to \chi)$ for every $x \leq y$. That is, $\phi \to \chi \not\sqsubseteq_{\text{RE}} \phi$. □

Recall from Section 2 that Lemma 1 contains some technical results about the faithful preorders. An application of this lemma to the faithful *modular weak partial orders* and their induced RE-orderings yields a very useful result. It shows that two wffs are equally entrenched if and only if their negations have the same minimal models, and for any two wffs ϕ and ψ, both of whom are not logically valid, ϕ is strictly more entrenched than ψ if and only if the minimal models of $\neg\psi$ are either strictly above the minimal models of $\neg\phi$, or form a strict subset of the minimal models of $\neg\phi$. As a consequence, two wffs ϕ and ψ are incomparable iff the minimal models of the negations of the two wffs are on the same level, the minimal models of $\neg\phi$ include a model of ψ, and the minimal models of $\neg\psi$ include a model of ϕ.

COROLLARY 2. *Let $\leq$ be a faithful modular weak partial order, and let $\sqsubseteq_{\text{RE}}$ be the RE-ordering induced from $\leq$.*

1. $\phi \equiv_{\sqsubseteq_{\text{RE}}} \psi$ *iff* $\text{Min}_{\leq}(\neg\psi) = \text{Min}_{\leq}(\neg\phi)$.
2. *If* $\not\models \phi$ *and* $\not\models \psi$ *then* $\phi \sqsubset_{\text{RE}} \psi$ *iff* $\text{Min}_{\leq}(\neg\psi) \subset \text{Min}_{\leq}(\neg\phi)$, *or for every* $y \in \text{Min}_{\leq}(\neg\psi)$ *and every* $x \in \text{Min}_{\leq}(\neg\phi)$, $x < y$.
3. $\phi \parallel_{\sqsubseteq_{\text{RE}}} \psi$ *iff* $\text{Min}_{\leq}(\neg\phi) \nsubseteq M(\neg\psi)$, $\text{Min}_{\leq}(\neg\psi) \nsubseteq M(\neg\phi)$, *and* $x \parallel_{leq} y$ *or* $x = y$ *for every* $y \in \text{Min}_{\leq}(\neg\psi)$ *and every* $x \in \text{Min}_{\leq}(\neg\phi)$.

Proof.

1. Follows easily from Lemma 1.
2. Suppose $\phi \sqsubset_{\text{RE}} \psi$, and suppose there is a $y \in \text{Min}_{\leq}(\neg\psi)$ and an $x \in \text{Min}_{\leq}(\neg\phi)$, such that $x \not< y$. From $\psi \not\sqsubseteq_{\text{RE}} \phi$ there is a $v \in \text{Min}_{\leq}(\neg\phi)$ such that $u \in M(\psi)$ for every $u \leq v$. So, for every $s \in \text{Min}_{\leq}(\neg\psi)$ and every $t \in \text{Min}_{\leq}(\neg\phi)$, $s \not< t$. Therefore, the minimal models of $\neg\phi$ and $\neg\psi$ lie on the same level. Now pick any $u \in \text{Min}_{\leq}(\neg\psi)$. By $\phi \sqsubseteq_{\text{RE}} \psi$, $u \in \text{Min}_{\leq}(\neg\phi)$. Furthermore, v is a minimal model of $\neg\phi$ that is not a minimal model of $\neg\psi$ and so $\text{Min}_{\leq}(\neg\psi) \subset \text{Min}_{\leq}(\neg\phi)$. Conversely, $\phi \sqsubset_{\text{RE}} \psi$ follows easily if $\text{Min}_{\leq}(\neg\psi) \subset \text{Min}_{\leq}(\neg\phi)$ or if $x < y$ for every $y \in \text{Min}_{\leq}(\neg\psi)$ and every $x \in \text{Min}_{\leq}(\neg\phi)$.
3. Suppose $\phi \parallel_{\sqsubseteq_{\text{RE}}} \psi$. So there is a $v \in \text{Min}_{\leq}(\neg\phi)$ such that $u \in M(\psi)$ for every $u \leq v$, and there is a $y \in \text{Min}_{\leq}(\neg\psi)$ such that $x \in M(\phi)$ for every $x \leq y$. The required result then follows from the fact that $v \parallel_{\leq} y$ and that $\leq$ is a modular weak partial order. The converse follows easily. □

A consequence of Corollaries 1 and 2 is that the RE-ordering associated with an EE-ordering $\sqsubseteq_{\text{EE}}$ maintains the ordering between the equivalence classes of wffs modulo $\sqsubseteq_{\text{EE}}$ but offers an exploded view of each of these equivalence classes. Figure 1 gives a graphical representation of this situation.

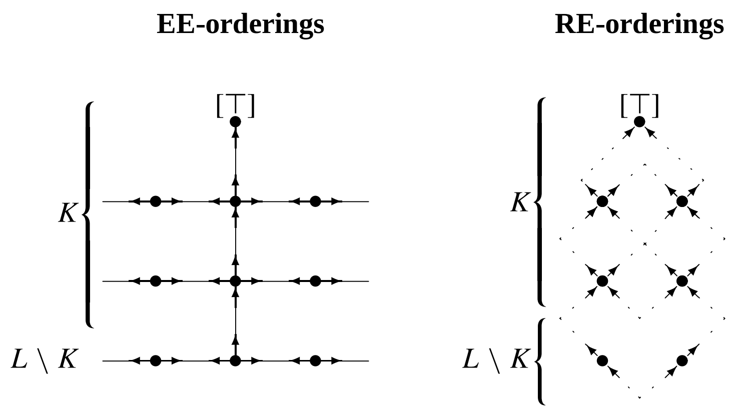

Figure 1. A graphical comparison of the EE-orderings and the RE-orderings.

We conclude this section by showing that the EE-orderings and the RE-orderings are interdefinable.

THEOREM 4. *Let $\sqsubseteq_{\mathrm{RE}}$ be an RE-ordering, and let $\sqsubseteq_{\mathrm{EE}}$ be its associated EE-ordering.*

1. *$\phi \sqsubseteq_{\mathrm{RE}} \psi$ iff $\vDash \psi$ or $\phi \sqsubset_{\mathrm{EE}} \psi$ or $\psi \sqsubset_{\mathrm{EE}} \phi \to \psi$, and*
2. *$\phi \sqsubseteq_{\mathrm{EE}} \psi$ iff $\phi \sqsubseteq_{\mathrm{RE}} \psi$ or $\phi \to \psi \not\sqsubseteq_{\mathrm{RE}} \phi$.*

Proof. Let $\leq$ be the faithful modular weak partial order that induces $\sqsubseteq_{\mathrm{RE}}$, and let $\preceq$ be the associated faithful total preorder that induces $\sqsubseteq_{\mathrm{EE}}$.

1. Suppose that $\phi \sqsubseteq_{\mathrm{RE}} \psi$. By part (2) of Proposition 3, $\phi \sqsubseteq_{\mathrm{EE}} \psi$. Suppose further that $\phi \not\sqsubset_{\mathrm{EE}} \psi$, i.e., $\psi \sqsubseteq_{\mathrm{EE}} \phi$, and that $\nvDash \psi$. Then $\nvDash \phi$ (by part (4) of Proposition 3), and by Corollary 1 it follows that for every $y \in \mathrm{Min}_{\preceq}(\neg\psi)$ and every $x \in \mathrm{Min}_{\preceq}(\neg\phi)$, $x \equiv_{\preceq} y$. Because $\nvDash \psi$ there is thus a $v \in \mathrm{Min}_{\leq}(\neg\psi)$ such that for every $u \prec v$, $u \in M(\phi) \cap M(\psi)$. Combined with $\phi \sqsubseteq_{\mathrm{RE}} \psi$ this means that for every $w \equiv_{\preceq} v$, $w \in M(\neg\psi) \cap M(\neg\phi)$ or $w \in M(\psi)$. Therefore, $z \in M(\phi \to \psi)$ for every $z \preceq v$, and so $\phi \to \psi \not\sqsubseteq_{\mathrm{EE}} \psi$, i.e., $\psi \sqsubset_{\mathrm{EE}} \phi \to \psi$. Conversely, if $\vDash \psi$ then $\phi \sqsubseteq_{\mathrm{RE}} \psi$ follows vacuously. If $\phi \sqsubset_{\mathrm{EE}} \psi$, i.e., $\psi \not\sqsubseteq_{\mathrm{EE}} \phi$, there is a $y \in \mathrm{Min}_{\preceq}(\neg\phi)$ such that $x \in M(\psi)$ for every $x \preceq y$. So $y < u$ for every $u \in \mathrm{Min}_{\leq}(\neg\psi)$ and therefore $\phi \sqsubseteq_{\mathrm{RE}} \psi$. Finally, suppose that $\psi \sqsubset_{\mathrm{EE}} \phi \to \psi$, i.e., $\phi \to \psi \not\sqsubseteq_{\mathrm{EE}} \psi$. Then there is a $y \in \mathrm{Min}_{\preceq}(\neg\psi)$ such that $x \in M(\phi \to \psi)$ for every $x \preceq y$, and so $\mathrm{Min}_{\leq}(\neg\psi) = \mathrm{Min}_{\preceq}(\neg\psi) \subseteq M(\neg\phi)$. So for every $v \in M(\neg\psi)$, there is a $u \in M(\neg\phi)$ such that $u \leq v$, i.e., $\phi \sqsubseteq_{\mathrm{RE}} \psi$.
2. Suppose that $\phi \sqsubseteq_{\mathrm{EE}} \psi$ and that $\phi \not\sqsubseteq_{\mathrm{RE}} \psi$. By $\phi \not\sqsubseteq_{\mathrm{RE}} \psi$ there is a $y \in \mathrm{Min}_{\leq}(\neg\psi)$ such that $x \in M(\phi)$ for every $x \leq y$. So $z \in M(\phi) \cap M(\psi)$ for every $z < y$, and from $\phi \sqsubseteq_{\mathrm{EE}} \psi$ it thus follows that there is a $v \equiv_{\preceq} y$ such that $v \in M(\neg\phi)$. So $u \in M(\phi \to \psi)$ for every $u \leq v$ and thus $\phi \to \psi \not\sqsubseteq_{\mathrm{RE}} \phi$.

Conversely, if $\phi \sqsubseteq_{\text{RE}} \psi$ then $\phi \sqsubseteq_{\text{EE}} \psi$ by part (2) of Proposition 3. And if $\phi \to \psi \not\sqsubseteq_{\text{RE}} \phi$ then there is a $y \in \text{Min}_{\leq}(\neg\phi)$ such that $x \in M(\phi \to \psi)$ for every $x \leq y$. Therefore, $z \in M(\phi) \cap M(\psi)$ for every $z < y$. So $u \not< y$ for every $u \in M(\neg\psi)$, from which $\phi \sqsubseteq_{\text{EE}} \psi$ follows easily. □

3.1. Postulates for Refined Entrenchment

In this section we present a description of the RE-orderings in terms of postulates, and give a representation theorem to prove that the postulates do indeed provide a characterisation of the RE-orderings. The postulates are given below.

$(RE1)$ $\sqsubseteq_{\text{RE}}$ is transitive.

$(RE2)$ If $\phi \models \psi$ then $\phi \sqsubseteq_{\text{RE}} \psi$.

$(RE3a)$ If $\phi, \psi \in K$ then $\phi \sqsubseteq_{\text{RE}} \phi \wedge \psi$, or $\psi \sqsubseteq_{\text{RE}} \phi \wedge \psi$,

or both $\phi \to \psi \not\sqsubseteq_{\text{RE}} \phi$ and $\psi \to \phi \not\sqsubseteq_{\text{RE}} \psi$.

$(RE3b)$ If $\phi \to \chi \sqsubseteq_{\text{RE}} \phi$ then $\phi \to \psi \sqsubseteq_{\text{RE}} \phi$ or $\psi \to \chi \sqsubseteq_{\text{RE}} \psi$.

$(RE3c)$ If $\phi \to \chi \sqsubseteq_{\text{RE}} \phi$ then $\phi \not\sqsubseteq_{\text{RE}} \psi$ or $\psi \to \chi \sqsubseteq_{\text{RE}} \psi$.

$(RE3d)$ If $\phi \to \chi \sqsubseteq_{\text{RE}} \phi$ then $\psi \not\sqsubseteq_{\text{RE}} \chi$ or $\phi \to \psi \sqsubseteq_{\text{RE}} \phi$.

$(RE4a)$ If $\phi \notin K$ and $\psi \in K$, then $\phi \sqsubset_{\text{RE}} \psi$.

$(RE4b)$ If $\phi, \psi \notin K$, then $\phi \sqsubseteq_{\text{RE}} \psi$ iff $K \cup \{\phi\} \models \psi$.

$(RE5)$ If $\phi \sqsubseteq_{\text{RE}} \psi$ for all ϕ, then $\models \psi$.

To a certain extent, the postulates for refined entrenchment follow the same pattern as the postulates for the EE-orderings, and this is reflected in the labelling scheme we use for the refined entrenchment postulates. $(RE1)$, $(RE2)$ and $(RE5)$ are identical to $(EE1)$, $(EE2)$, and $(EE5)$, respectively. And while $(RE3a)$ bears a vague resemblance to $(EE3)$, it is a bit more difficult to describe the intuition associated with $(RE3b)$, $(RE3c)$ and $(RE3d)$. Technically though, they seem to be necessary for a complete description of the relationship between the wffs in K. $(EE4)$ gives a complete description of how an EE-ordering treats the wffs that are not in K, while the handling of such wffs by the RE-orderings are described by the two independent postulates, $(RE4a)$ and $(RE4b)$. $(RE4a)$ describes the relationship between wffs in K and wffs not in K, while $(RE4b)$ is a prescription for the treatment of any two wffs, neither of which are in K. To obtain the desired representation theorem, we need the following two lemmas.

LEMMA 3. *If $\sqsubseteq$ is a relation on L that satisfies* $(RE1$–$RE5)$ *then the relation* $\sqsubseteq_{\text{EE}}$ *defined as: $\phi \sqsubseteq_{\text{EE}} \psi$ iff $\phi \sqsubseteq \psi$ or $\phi \to \psi \not\sqsubseteq \phi$, satisfies* $(EE1$–$EE5)$.

Proof. For $(EE1)$, suppose that $\phi \sqsubseteq_{\text{EE}} \psi$ and $\psi \sqsubseteq_{\text{EE}} \chi$. That is, $\phi \sqsubseteq \psi$ or $\phi \to \psi \not\sqsubseteq \phi$, and $\psi \sqsubseteq \chi$ or $\psi \to \chi \not\sqsubseteq \psi$. This can be divided into four cases: (i) $\phi \sqsubseteq \psi$ and $\psi \sqsubseteq \chi$, (ii) $\phi \sqsubseteq \psi$ and $\psi \to \chi \not\sqsubseteq \psi$, (iii) $\psi \sqsubseteq \chi$ and $\phi \to \psi \not\sqsubseteq \phi$, and (iv) $\phi \to \psi \not\sqsubseteq \phi$ and $\psi \to \chi \not\sqsubseteq \psi$. For (i), $\phi \sqsubseteq \chi$ follows from $(RE1)$. For (ii), (iii), and (iv), $\phi \to \chi \not\sqsubseteq \phi$ follows from $(RE3c)$, $(RE3d)$, and $(RE3b)$ respectively. So in all four cases, either $\phi \sqsubseteq \chi$ or $\phi \to \chi \not\sqsubseteq \phi$. That is, $\phi \sqsubseteq_{\text{EE}} \chi$. $(EE2)$ follows from $(RE2)$ and $(EE3)$ follows from $(RE2)$, $(RE3a)$, $(RE4a)$, and $(RE4b)$. For $(EE4)$, suppose that $K \neq L$, and let $\phi \notin K$. Assume there is a ψ such that $\phi \not\sqsubseteq_{\text{EE}} \psi$. That is, $\phi \not\sqsubseteq \psi$ and $\phi \to \psi \sqsubseteq \phi$. By $(RE4a)$ $\psi \notin K$, and so, by $(RE4b)$, $K \cup \{\phi \to \psi\} \vDash \phi$. But this means $\phi \in K$; a contradiction. Conversely, suppose that $\phi \in K$. So $\neg\phi \notin K$, and $\neg\phi \sqsubset \phi$ by $(RE4a)$. And since $\phi \to \neg\phi \equiv \neg\phi$, we have that $\phi \not\sqsubseteq \neg\phi$ and $\phi \to \neg\phi \sqsubseteq \phi$. That is, $\phi \not\sqsubseteq_{\text{EE}} \neg\phi$. For $(EE5)$, suppose that $\nvDash \psi$. By $(RE5)$, $\top \not\sqsubseteq \psi$ and by $(RE2)$, $\top \to \psi \sqsubseteq \top$. That is, $\top \not\sqsubseteq_{\text{EE}} \psi$. □

LEMMA 4. *Let $\sqsubseteq$ be a relation on L that satisfies $(RE1$–$RE5)$. If $\sqsubseteq_{\text{EE}}$ is defined as: $\phi \sqsubseteq_{\text{EE}} \psi$ iff $\phi \sqsubseteq \psi$ or $\phi \to \psi \not\sqsubseteq \phi$, and $\sqsubseteq_{\text{RE}}$ is defined as as: $\phi \sqsubseteq_{\text{RE}} \psi$ iff $\vDash \psi$ or $\phi \sqsubset_{\text{EE}} \psi$ or $\psi \sqsubset_{\text{EE}} \phi \to \psi$, then $\sqsubseteq = \sqsubseteq_{\text{RE}}$.*

Proof. By Lemma 3, $\sqsubseteq_{\text{EE}}$ is an EE-ordering, and thus a total preorder. By keeping in mind that $\phi \sqsubset_{\text{EE}} \psi$ iff $\psi \not\sqsubseteq_{\text{EE}} \phi$, noting that $(\phi \to \psi) \to \psi \equiv \phi \vee \psi$, and combining the definitions of $\sqsubseteq_{\text{EE}}$ and $\sqsubseteq_{\text{RE}}$, it suffices to show that

$$\phi \sqsubseteq \psi \ \text{ iff } \begin{cases} \vDash \psi, \text{ or} \\ \psi \not\sqsubseteq \phi \text{ and } \psi \to \phi \sqsubseteq \psi, \text{ or} \\ \phi \to \psi \not\sqsubseteq \psi \text{ and } \phi \vee \psi \sqsubseteq \phi \to \psi. \end{cases}$$

So suppose that $\phi \sqsubseteq \psi$, $\nvDash \psi$, and either $\psi \sqsubseteq \phi$ or $\psi \to \phi \not\sqsubseteq \psi$. We have to show that $\phi \to \psi \not\sqsubseteq \psi$ and $\phi \vee \psi \sqsubseteq \phi \to \psi$. Assume that $\phi \to \psi \sqsubseteq \psi$. There are two cases. Either $\psi \sqsubseteq \phi$ or $\psi \to \phi \not\sqsubseteq \psi$. In the former case, $\phi \to \psi \sqsubseteq \psi \sqsubseteq \phi$. By $(RE3c)$ it thus follows that $\phi \not\sqsubseteq \psi$ or $\psi \to \psi \sqsubseteq \psi$, contradicting $\phi \sqsubseteq \psi$ and $\nvDash \psi$ combined with $(RE5)$. In the latter case, note that $\phi \sqsubseteq \psi \sqsubseteq \phi \to \psi$ by $(RE2)$, and since $(\phi \to \psi) \to \phi \equiv \phi$, $(\phi \to \psi) \to \phi \sqsubseteq \phi \to \psi$. By $(RE3c)$ we then have that $\phi \to \psi \not\sqsubseteq \psi$, or $\psi \to \phi \sqsubseteq \psi$; a contradiction. So we have shown that $\phi \to \psi \not\sqsubseteq \psi$. Now assume that $\phi \vee \psi \not\sqsubseteq \phi \to \psi$. By $(RE2)$, $\phi \sqsubseteq \psi \sqsubseteq \phi \to \psi$. And since $\phi \equiv (\phi \to \psi) \to \phi$, $(\phi \to \psi) \to \phi \sqsubseteq \phi \to \psi$. By $(RE3b)$ it then follows that $(\phi \to \psi) \to \psi \sqsubseteq \phi \to \psi$, or $\psi \to \phi \sqsubseteq \psi$. And since $(\phi \to \psi) \to \psi \equiv \phi \vee \psi$, it has to be the case that $\psi \to \phi \sqsubseteq \psi$. But since we have, by supposition, that $\psi \sqsubseteq \phi$ or $\psi \to \phi \not\sqsubseteq \psi$, this means that $\psi \sqsubseteq \phi$. From $\psi \to \phi \sqsubseteq \psi$ it also follows by $(RE3c)$ that $\psi \not\sqsubseteq \phi$ or $\phi \to \phi \sqsubseteq \phi$. So $\phi \to \phi \sqsubseteq \phi$, and by $(RE5)$, $\vDash \phi$. But this contradicts $\nvDash \psi$, $\phi \sqsubseteq \psi$, and $(RE5)$. □

We are now in a position to prove that the postulates given above provide a characterisation of the RE-orderings.

THEOREM 5. *A relation $\sqsubseteq$ on L satisfies $(RE1–RE5)$ iff there is a faithful modular weak partial order that induces $\sqsubseteq$.*

Proof. Let $\sqsubseteq$ be a relation on L that satisfies $(RE1–RE5)$. Now define a relation $\sqsubseteq_{\mathrm{EE}}$ on L as follows: $\phi \sqsubseteq_{\mathrm{EE}} \psi$ iff $\phi \sqsubseteq \psi$ or $\phi \to \psi \not\sqsubseteq \phi$. By Lemma 3, $\sqsubseteq_{\mathrm{EE}}$ is an EE-ordering, and by Theorem 3 there is thus a faithful total preorder $\preceq$ from which $\sqsubseteq_{\mathrm{EE}}$ can be induced. By Theorem 4, the faithful modular weak partial order associated with $\preceq$ induces the RE-ordering $\sqsubseteq_{\mathrm{RE}}$, defined as follows: $\phi \sqsubseteq_{\mathrm{RE}} \psi$ iff $\models \psi$ or $\phi \sqsubset_{\mathrm{EE}} \psi$ or $\psi \sqsubset_{\mathrm{EE}} \phi \to \psi$. And by Lemma 4, $\sqsubseteq$ and $\sqsubseteq_{\mathrm{RE}}$ are identical. The converse follows from Proposition 3 and Lemma 2. □

3.2. Refined Entrenched and AGM Contraction

Just as for the EE-orderings, the RE-orderings and AGM contraction are interdefinable.

THEOREM 6. *Let $\leq$ be a faithful modular weak partial order, $\sqsubseteq_{\mathrm{RE}}$ the RE-ordering induced by $\leq$, and $-$ the contraction function constructed from $\leq$.*

1. *The contraction function $-$ can be obtained from $\sqsubseteq_{\mathrm{RE}}$ as follows:*

$$\psi \in K - \phi \text{ iff } \psi \in K \text{ and } \begin{cases} \phi \not\sqsubseteq_{\mathrm{RE}} \psi \to \phi, \text{ or} \\ \phi \notin K. \end{cases}$$

2. *The RE-ordering $\sqsubseteq_{\mathrm{RE}}$ can be obtained from $-$ as follows:*

$$\phi \sqsubseteq_{\mathrm{RE}} \psi \text{ iff } \begin{cases} \models \psi, \text{ or} \\ \not\models \phi \text{ and } \psi \in K - \phi \wedge \psi, \text{ or} \\ \phi \to \psi \in K - \psi. \end{cases}$$

Proof.

1. We only consider the case where $\phi, \psi \in K$, the remaining cases are trivial. It suffices to show that $\mathrm{Min}_{\leq}(\neg\phi) \subseteq M(\psi)$ iff $\phi \not\sqsubseteq_{\mathrm{RE}} \psi \to \phi$. Now $\mathrm{Min}_{\leq}(\neg\phi) \subseteq M(\psi)$ iff $y \Vdash \psi$ for every $y \in \mathrm{Min}_{\leq}(\neg\phi)$, iff for every $y \in M(\neg\phi)$ there is an $x \in M(\neg\phi) \cap M(\psi)$ such that $x \leq y$, iff for every $y \in M(\neg\phi)$ there is an $x \in M(\neg(\psi \to \phi))$ such that $x \leq y$, iff $\psi \to \phi \sqsubseteq_{\mathrm{RE}} \phi$, iff $\phi \not\sqsubseteq_{\mathrm{RE}} \psi \to \phi$.
2. Follows from Theorems 2 and 4. □

As expected, the construction of a contraction function from an EE-ordering, as described in Theorem 2, and the construction of a contraction function from its associated RE-ordering, as described above, both yield the same contraction function. This follows easily from Propositions 1 and 2, and Theorem 4.

In the case of a finitely generated propositional language, the definition of AGM contraction in terms of the RE-orderings can be simplified considerably. To see why, consider a faithful modular weak partial order $\leq$ on the interpretations of such a finite L, and let $\sqsubseteq_{\mathrm{RE}}$ be the RE-ordering induced by $\leq$. From part (10) of Proposition 3, it follows that for every ϕ, there is a $\psi \in [\phi]_{\sqsubseteq_{\mathrm{RE}}}$ such that $\chi \vDash \psi$ for every $\chi \in [\phi]_{\sqsubseteq_{\mathrm{RE}}}$. That is, every equivalence class $[\phi]_{\sqsubseteq_{\mathrm{RE}}}$ contains a logically weakest wff. We use $\ulcorner\phi\urcorner_{\mathrm{RE}}$ to denote a canonical representative of the logically weakest wffs in $[\phi]_{\sqsubseteq_{\mathrm{RE}}}$, and show below that if $\phi, \psi \in K$, then $\psi \in K - \phi$ iff $\psi \to \phi \vDash \ulcorner\phi\urcorner_{\mathrm{RE}}$. That is, if ϕ is in K, then checking whether a wff $\psi \in K$ is retained in $K - \phi$ is a matter of checking whether $\psi \to \phi$ entails a logically weakest wff in $[\phi]_{\sqsubseteq_{\mathrm{RE}}}$.

PROPOSITION 4. *Let L be a finitely generated propositional language, $\leq$ a faithful modular weak partial order, $-$ the AGM contraction function constructed from $\leq$, and $\sqsubseteq_{\mathrm{RE}}$ the RE-ordering induced by $\leq$. If $\phi, \psi \in K$ then $\psi \in K - \phi$ iff $\psi \to \phi \vDash \ulcorner\phi\urcorner_{\sqsubseteq_{\mathrm{RE}}}$ (where $\ulcorner\phi\urcorner_{\mathrm{RE}}$ is a canonical representative of the logically weakest wffs in $[\phi]_{\sqsubseteq_{\mathrm{RE}}}$).*

Proof. By Theorem 6, if $\phi, \psi \in K$, then $\psi \in K - \phi$ iff $\phi \not\sqsubseteq_{\mathrm{RE}} \psi \to \phi$. Since $\phi \vDash \psi \to \phi$, it follows from part (3) of Proposition 3 that $\phi \sqsubseteq_{\mathrm{RE}} \psi \to \phi$, and this result can thus be rewritten as follows: if $\phi, \psi \in K$, then $\psi \in K - \phi$ iff $\psi \to \phi \in [\phi]_{\sqsubseteq_{\mathrm{RE}}}$. Now suppose that $\phi, \psi \in K$. If $\psi \in K - \phi$, then $\psi \to \phi \in [\phi]_{\sqsubseteq_{\mathrm{RE}}}$, and since $\ulcorner\phi\urcorner_{\sqsubseteq_{\mathrm{RE}}}$ is logically weaker than every wff in $[\phi]_{\sqsubseteq_{\mathrm{RE}}}$, $\psi \to \phi \vDash \ulcorner\phi\urcorner_{\sqsubseteq_{\mathrm{RE}}}$. Conversely, if $\psi \to \phi \vDash \ulcorner\phi\urcorner_{\sqsubseteq_{\mathrm{RE}}}$ then, by part (3) of Proposition 3, $\psi \to \phi \sqsubseteq_{\mathrm{RE}} \ulcorner\phi\urcorner_{\sqsubseteq_{\mathrm{RE}}}$. Furthermore, since $\ulcorner\phi\urcorner_{\sqsubseteq_{\mathrm{RE}}} \in [\phi]_{\sqsubseteq_{\mathrm{RE}}}$, we get that $\ulcorner\phi\urcorner_{\sqsubseteq_{\mathrm{RE}}} \sqsubseteq_{\mathrm{RE}} \phi$, and so, by the transitivity of $\sqsubseteq_{\mathrm{RE}}$, $\psi \to \phi \sqsubseteq_{\mathrm{RE}} \phi$. And because $\phi \vDash \psi \to \phi$, it follows from part (3) of Proposition 3 that $\phi \sqsubseteq_{\mathrm{RE}} \psi \to \phi$. Thus $\psi \to \phi \in [\phi]_{\sqsubseteq_{\mathrm{RE}}}$, and it follows from the result above that $\psi \in K - \phi$. □

Observe that the property described in Proposition 4 does not hold for the EE-orderings. The reason for this is that an equivalence class $[\phi]_{\sqsubseteq_{\mathrm{EE}}}$, where $\sqsubseteq$ is an EE-ordering, may contain logically weakest elements that are not logically equivalent. As a result, it does not make sense to talk about a canonical representative of the logically weakest wffs in such an equivalence class. The example below describes such a situation.

EXAMPLE 1. Let L be the propositional language generated by the two atoms p and q, let $K = Cn(\{p\})$, and define the binary relation $\sqsubseteq_{\mathrm{EE}}$ as follows:

$$\phi \sqsubseteq_{\mathrm{EE}} \psi \text{ iff } \begin{cases} \psi \in L \text{ if } \phi \notin K, \\ \psi \in K \text{ if } \phi \in K \setminus Cn(\top), \\ \vDash \psi \text{ if } \vDash \phi. \end{cases}$$

It is readily verified that $\sqsubseteq_{\mathrm{EE}}$ is an EE-ordering with respect to K. Now observe that the equivalence class $[p]_{\sqsubseteq_{\mathrm{EE}}}$ contains the wffs $p \vee q$ and $p \vee \neg q$, but nothing that is logically weaker than either of these.

4. A Comparison with Related Approaches

Refined entrenchment is certainly not the first attempt at a definition of entrenchment without the requirement that all wffs be comparable. Two notable proposals are those of Lindström and Rabinowicz (1991), and Rott (1992). The entrenchment orderings of Lindström and Rabinowicz (which we refer to as the *LR-orderings*) are generalisations of the EE-orderings. They adopt the same postulates, except for the replacement of $(EE3)$ by the following weaker postulate:

$(EE3')$ If $\phi \sqsubseteq \psi$ and $\phi \sqsubseteq \chi$ then $\phi \sqsubseteq \psi \wedge \chi$.

Lindström and Rabinowicz point out that the EE-orderings form a strict subset of their entrenchment orderings. It is easily verified that the RE-orderings also satisfy $(EE3')$.

PROPOSITION 5. *Every RE-ordering* $\sqsubseteq_{\mathrm{RE}}$ *satisfies* $(EE3')$.

Proof. Suppose $\phi \sqsubseteq_{\mathrm{RE}} \psi$ and $\phi \sqsubseteq_{\mathrm{RE}} \chi$, let $\leq$ be the faithful modular weak partial order from which $\sqsubseteq_{\mathrm{RE}}$ is induced, and pick a $y \in M(\neg(\psi \wedge \chi))$. So $y \in M(\neg\psi)$ or $y \in M(\neg\chi)$. In the former case it follows from $\phi \sqsubseteq_{\mathrm{RE}} \psi$ that there is an $x \in M(\neg\phi)$ such that $x \leq y$. And similarly for the latter case and $\phi \sqsubseteq_{\mathrm{RE}} \chi$. So $\phi \sqsubseteq_{\mathrm{RE}} \psi \wedge \chi$. □

Since the LR-orderings require that all the wffs not in K be equally entrenched, the RE-orderings do not qualify as instances of the LR-orderings. However, the RE-orderings conform to the conditions imposed by the LR-orderings on the wffs in K. In this sense, there is an LR-ordering corresponding to every RE-ordering. On the other hand, the following example shows that some LR-orderings do not correspond to any RE-ordering, even when we restrict ourselves to just the wffs in K.

EXAMPLE 2. Consider the propositional language L generated by the two atoms p and q, let $K = Cn(\{p \wedge q\})$, and consider the LR-ordering $\sqsubseteq_{LR}$ defined as follows:

$$\phi \sqsubseteq_{LR} \psi \text{ iff } \begin{cases} \psi \in L \text{ if } \phi \notin K, \\ p \wedge q \vDash \psi \text{ if } \phi \equiv p \wedge q \text{ or } \phi \equiv p \leftrightarrow q \text{ or } \phi \equiv p \text{ or} \\ \phi \equiv q \text{ or } \phi \equiv \neg p \vee q, \\ p \vee q \vDash \psi \text{ if } \phi \equiv p \vee q, \\ p \vee \neg q \vDash \psi \text{ if } \phi \equiv p \vee \neg q, \\ \vDash \psi \text{ if } \vDash \phi. \end{cases}$$

Figure 2 contains a graphical representation of $\sqsubseteq_{LR}$. An inspection of Figure 2 reveals that $\sqsubseteq_{LR}$ is indeed an LR-ordering, but it can be verified that the part of $\sqsubseteq_{LR}$ restricted to the elements of K does not coincide with the restriction of any RE-ordering $\sqsubseteq_{\mathrm{RE}}$ to K. Furthermore, $\sqsubseteq_{LR}$ is not an EE-ordering either, since it is clearly not a total preorder.

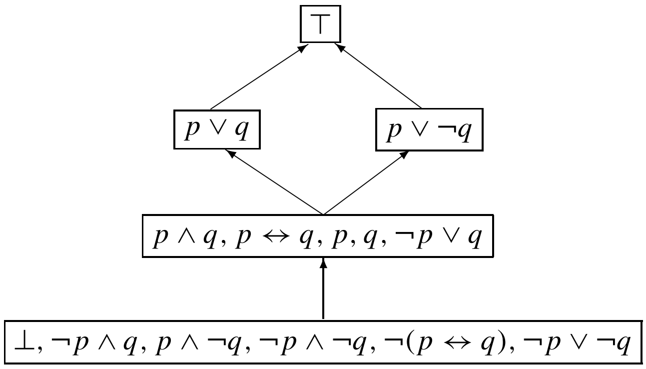

Figure 2. A graphical representation of the LR-ordering used in Example 2. The ordering is obtained from the reflexive transitive closure of the relation determined by the arrows. Every wff in the figure is a canonical representative of the set of wffs that are logically equivalent to it.

Rott (1992) takes the view that it is more natural to consider *strict* relations on wffs and argues that the EE-orderings should be seen as converse complements of such strict relations (or equivalently, that these strict relations be obtained as the converse complements of the EE-orderings).* He proposes the following set of postulates:

$(GEE1)$ $\top \not\sqsubset \top$.

$(GEE2\uparrow)$ If $\phi \sqsubset \psi$ and $\psi \vDash \chi$ then $\phi \sqsubset \chi$.

$(GEE2\downarrow)$ If $\phi \sqsubset \psi$ and $\chi \vDash \phi$ then $\chi \sqsubset \psi$.

$(GEE3\uparrow)$ If $\phi \sqsubset \psi$ and $\phi \sqsubset \chi$ then $\phi \sqsubset \psi \wedge \chi$.

$(GEE3\downarrow)$ If $\phi \wedge \psi \sqsubset \psi$ then $\phi \sqsubset \psi$.

A relation satisfying these postulates is known as a generalised epistemic entrenchment ordering (a GEE-ordering). Rott shows that the converse complements of the EE-orderings form a strict subset of the GEE-orderings.

The GEE-orderings, as defined above, are not subject to analogues of the minimality and maximality conditions imposed on the EE-orderings. (Minimality for the EE-orderings is the requirement that all the wffs not in K be equally entrenched, and less entrenched than all the wffs in $\dot{K}$, while maximality is the condition that the logically valid wffs be equally entrenched, and more entrenched than the remaining wffs.) The following four supplementary postulates for generalised epistemic entrenchment are intended to serve as such analogues:

* A relation S is the converse complement of a binary relation R on a set X iff for every $x, y \in X$, $(x, y) \in S$ iff $(y, x) \notin R$.

$(GEE4)$ If $K \neq L$ then $\bot \sqsubset \phi$ iff $\phi \in K$.

$(GEE4')$ If $\phi \notin K$ and $\psi \in K$ then $\phi \sqsubset \psi$.

$(GEE5)$ If $\nvDash \phi$ then $\phi \sqsubset \top$.

$(GEE5')$ If $\phi \sqsubset \top$ and $\psi \not\sqsubset \top$ then $\phi \sqsubset \psi$.

It is easily verified that the converse complements of the EE-orderings satisfy these four postulates as well.

Since the EE-orderings are total preorders, taking the converse complement of an EE-ordering $\sqsubseteq_{\mathrm{EE}}$ is the same as taking its strict version $\sqsubset_{\mathrm{EE}}$. In the case of the RE-orderings, this is not the case, though. One way to obtain a comparison of the RE-orderings with the GEE-orderings is to check whether the RE-orderings satisfy the following translations of the GEE postulates into assertions about the converse complements of the GEE-orderings:

$(CGEE1)$ $\top \sqsubseteq \top$.

$(CGEE2\uparrow)$ If $\chi \sqsubseteq \phi$ and $\psi \vDash \chi$ then $\psi \sqsubseteq \phi$.

$(CGEE2\downarrow)$ If $\psi \sqsubseteq \chi$ and $\chi \vDash \phi$ then $\psi \sqsubseteq \phi$.

$(CGEE3\uparrow)$ If $\psi \wedge \chi \sqsubseteq \phi$ then $\psi \sqsubseteq \phi$ or $\chi \sqsubseteq \phi$.

$(CGEE3\downarrow)$ If $\psi \sqsubseteq \phi$ then $\psi \sqsubseteq \phi \wedge \psi$.

$(CGEE4)$ If $K \neq L$ then $\phi \sqsubseteq \bot$ iff $\phi \notin K$.

$(CGEE4')$ If $\psi \in K$ and $\psi \sqsubseteq \phi$ then $\phi \in K$.

$(CGEE5)$ If $\top \sqsubseteq \phi$ then $\vDash \phi$.

$(CGEE5')$ If $\top \sqsubseteq \psi$ and $\psi \sqsubseteq \phi$ then $\top \sqsubseteq \phi$.

It is easily verified that the RE-orderings satisfy $(CGEE1)$, $(CGEE2\uparrow)$, $(CGEE2\downarrow)$, the three postulates regarded by Rott as minimal conditions of rationality for every relation designed to formalise the concept of epistemic entrenchment. Furthermore, they also satisfy $(CGEE3\downarrow)$, $(CGEE4')$, $(CGEE5)$ and $(CGEE5')$, but do not satisfy $(CGEE3\uparrow)$ and $(CGEE4)$. They do satisfy the left-to-right direction of $(CGEE4)$, though.

PROPOSITION 6. *Every RE-ordering* $\sqsubseteq_{\mathrm{RE}}$ *satisfies* $(CGEE1)$, $(CGEE2\uparrow)$, $(CGEE2\downarrow)$, $(CGEE3\downarrow)$, $(CGEE4')$, $(CGEE5)$ *and* $(CGEE5')$. *Furthermore, it does not necessarily satisfy* $(CGEE3\uparrow)$ *and* $(CGEE4)$, *but it does satisfy the left-to-right direction of* $(CGEE4)$.

Proof. Let $\leq$ be a faithful modular weak partial order that induces $\sqsubseteq_{\text{RE}}$. $(CGEE1)$ follows from $(RE2)$, $(CGEE2\ \uparrow)$ and $(CGEE2\ \downarrow)$ both from $(RE1)$ and $(RE2)$. For $(CGEE3\ \downarrow)$, suppose that $\psi \sqsubseteq_{\text{RE}} \phi$ and pick any $y \in M(\neg(\phi \wedge \psi))$. So $y \in M(\neg\psi)$ or $y \in M(\neg\phi)$. We have to show that there is an $x \leq y$ such that $x \in M(\neg\psi)$. If $y \in M(\neg\psi)$, this follows from the reflexivity of $\leq$, and if $y \in M(\neg\phi)$, it follows from the fact that $\psi \sqsubseteq_{\text{RE}} \phi$. $(CGEE4')$ follows from $(RE4a)$, $(CGEE5)$ from $(RE2)$ and $(RE5)$, and $(CGEE5')$ follows from $(RE1)$.

To show that the RE-orderings do not always satisfy $(CGEE3\ \uparrow)$, let $K = Cn(\{p \leftrightarrow q\})$ and consider the RE-ordering $\sqsubseteq_{\text{RE}}$, with respect to K, which is defined as follows:

$$\phi \sqsubseteq_{\text{RE}} \psi \text{ iff } \begin{cases} \psi \in L \text{ if } \phi \notin K, \\ \psi \in K \text{ if } \phi \equiv p \leftrightarrow q, \\ p \to q \vDash \psi \text{ if } \phi \equiv p \to q, \\ q \to p \vDash \psi \text{ if } \phi \equiv q \to p, \\ \vDash \psi \text{ if } \vDash \phi. \end{cases}$$

It is readily verified that $\sqsubseteq_{\text{RE}}$ is indeed an RE-ordering. By letting $\phi = p \leftrightarrow q$, $\psi = q \leftarrow q$, $\chi = p \leftarrow q$, and observing that $\psi \wedge \chi \equiv \phi$, we see that $\sqsubseteq_{\text{RE}}$ violates $(CGEE3\ \uparrow)$.

To show that the RE-orderings do not always satisfy $(CGEE4)$, it is sufficient to observe that the entailment relation $\vDash$ is an RE-ordering with respect to the belief set $Cn(\top)$. Finally, that every RE-ordering satisfies the left-to-right direction of $(CGEE4)$ follows from $(RE4a)$. $\square$

As observed above, the converse complement of an EE-ordering is the same as its strict version. It might therefore be instructive to determine whether or not the strict versions of the RE-orderings are instances of the GEE-orderings. It turns out that the strict RE-orderings satisfy $(GEE1)$, $(GEE2\ \uparrow)$, $(GEE2\ \downarrow)$, $(GEE4')$, $(GEE5)$ and $(GEE5')$, but do not always satisfy $(GEE3\ \uparrow)$, $(GEE3\ \downarrow)$ and $(GEE4)$, although they do satisfy the right-to-left direction of $(GEE4)$.

PROPOSITION 7. *The strict version* $\sqsubset_{\text{RE}}$ *of an RE-ordering satisfies* $(GEE1)$, $(GEE2\ \uparrow)$ *and* $(GEE2\ \downarrow)$, $(GEE4')$, $(GEE5)$ *and* $(GEE5')$, *but does not necessarily satisfy* $(GEE3\ \uparrow)$, $(GEE3\ \downarrow)$ *and* $(GEE4)$. *It does satisfy the right-to-left direction of* $(GEE4)$, *though.*

Proof. $(GEE1)$ is trivial. $(GEE2\ \uparrow)$ and $(GEE2\ \downarrow)$ both follow from $(RE1)$ and $(RE2)$. $(GEE4')$ follows from $(RE4a)$, and both $(GEE5)$ and $(GEE5')$ follow from $(RE2)$ and $(RE5)$. To show that $(GEE3\ \uparrow)$ and $(GEE3\ \downarrow)$ do not always hold, consider the propositional language L generated by the two atoms p and q. We represent the four interpretations of L by sequences 0s and 1s, where the first digit in a sequence indicates the truth value of p (1 denoting truth and 0 denoting falsity) and the second digit the truth value of q. So $U = \{11, 10, 01, 00\}$.

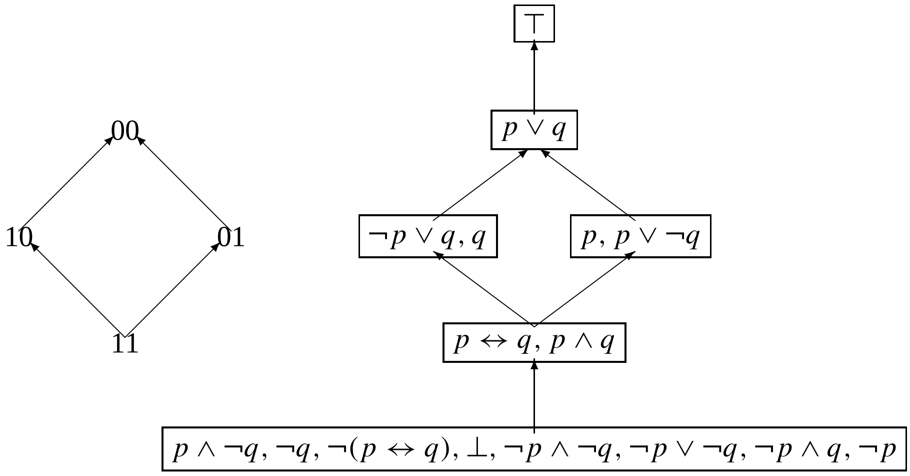

Figure 3. A graphical representation of the faithful modular weak partial order and the RE-ordering induced by it, used in Proposition 7. Every wff indicated in the RE-ordering is a canonical representative of the equivalence class of wffs logically equivalent to it.

Let $K = Cn(\{p \wedge q\})$, and let $\leq$ be the faithful modular weak partial order defined as follows:

$$\leq = \{(x, x) \mid x \in U\} \cup \{(11, y) \mid y \in U\} \cup \{(10, 00), (01, 00)\}.$$

Figure 3 contains a graphical representation of $\leq$ and the RE-ordering induced by $\leq$. An inspection of Figure 3 shows that $(GEE3\uparrow)$ is violated by taking ϕ as $p \leftrightarrow q$, ψ as p, and χ as q. $(GEE3\downarrow)$ is violated by taking ϕ as p and ψ as q.

Finally, $(RE4b)$ ensures that the left-to-right direction of $(GEE4)$ does not always hold, and $(RE4a)$ ensures that the right-to-left direction holds. □

The results above seem to suggest that the GEE-orderings and the RE-orderings have quite different intuitions associated with them. Whereas the GEE-orderings constitute a proper generalisation of the EE-orderings, the RE-orderings should be seen as refined alternatives to the EE-orderings. This also becomes clear when the link with contraction is investigated. Rott applies the construction of Theorem 2 to the GEE-orderings, and thus obtains a set of contraction functions that is a strict superset of AGM contraction. In contrast, Theorem 6 applies a different construction to the RE-orderings to obtain precisely the set of AGM contraction functions.

We conclude this discussion with a few thoughts on the comparability (or lack thereof) of the wffs not in K. Although the RE-orderings are able to distinguish between the entrenchment of wffs not in K (unlike the EE-orderings), this ability is little more than a byproduct of the power-construction used to induce the RE-orderings from the faithful modular weak partial orders, and does not seem to express a genuine difference in the entrenchment of such wffs. One way to obtain an appropriate entrenchment ordering on the wffs not in K, is to apply an idea of

Rabinowicz (1995) to the RE-orderings. Rabinowicz's suggestion concerns the EE-orderings and their dual Grove orderings (1988). Gärdenfors (1988) showed that Grove's orderings (1988) can be defined in terms of the EE-orderings as follows: $\phi \sqsubseteq_G \psi$ iff $\neg\phi \sqsubseteq_{\text{EE}} \neg\psi$. Rabinowicz proposes to combine an EE-ordering and its dual Grove ordering by applying the EE-ordering to the wffs in K, and the Grove ordering to the wffs not in K (and then placing the wffs in K all strictly above the wffs not in K). Since the definition of the Grove orderings in terms of the EE-orderings can also be applied to the RE-orderings, thus obtaining a set of dual Grove-like orderings, the Rabinowicz proposal can be extended to apply to the RE-orderings as well. Another suggestion by Rabinowicz (1995) is to modify the EE-orderings by doing away with $(EE4)$, applying $(EE3)$ to all the wffs in L, and to introduce a postulate corresponding to $(RE4a)$. This idea is easily applicable to the RE-orderings as well. Simply do away with $(RE4b)$, and modify $(RE3a)$ to apply to all the wffs in L. We shall leave an investigation of such modifications for future research.

5. Conclusion

We have presented a set of refined versions of the epistemic entrenchment orderings of Gärdenfors and Makinson, that allows for the introduction of a notion of irrelevance of wffs with respect to one another. Refined entrenchment was defined semantically, but a characterisation in terms of postulates was also provided. It was shown that refined entrenchment are interdefinable with epistemic entrenchment as defined by Gärdenfors and Makinson, as well as with AGM theory contraction. Furthermore, refined entrenchment was compared to the related approaches of Lindström and Rabinowicz (1991), and Rott (1992). While both these approaches also present versions of epistemic entrenchment in which wffs may be incomparable, they are both intended as generalisations of the entrenchment orderings of Gärdenfors and Makinson. In contrast, the refined entrenchment orderings are presented as refined alternatives. Indeed, it is easily seen that there is a one-to-one correspondence between the orderings of Gärdenfors and Makinson and the refined entrenchment orderings.

To conclude, we remark on possible future research. Some brief remarks on refined entrenchment and its treatment of the wffs not in the belief set K were made in section 4. But a proper investigation of exactly how this is to be done still needs to be undertaken. Another area in which refined entrenchment may be constructively applied involves the investigation of theory contraction operations that satisfy all the AGM contraction postulates, except for the controversial recovery postulate, $(K - 5)$. Makinson (1987) refers to such operations as withdrawal operations.) A first step in that direction is the description in Meyer et al. (1998) of the relationship between refined entrenchment and systematic withdrawal. But the links between refined entrenchment and other withdrawal operations, such as Rott and Pagnucco's severe withdrawal (1999) and Fermé's semi-contraction (1998), still need to be investigated.

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