# On the semantics of combination operations

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*ABSTRACT. Intelligent agents are often faced with the problem of trying to combine possibly conflicting pieces of information obtained from different sources into a coherent view of the world. We propose a framework for the modelling of such* combination *operations with roots in the work of Spohn [Spo88, Spo91]. We construct a number of combination operations and we measure them against various properties that such operations ought to satisfy. We conclude by discussing the connection between combination operations and the use of infobases [Mey99, MLH00].*
*KEYWORDS: Knowledge representation, belief revision, combination operations, merging, knowledge bases, epistemic states.*

## 1 Introduction

To be able to operate in its environment an intelligent agent must have a coherent view of the world. This demand is often complicated by the fact that such agents receive conflicting pieces of information from different sources. This issue is usually addressed by considering the *merging* of possibly inconsistent knowledge bases [BI84, Lin96, BKM91, BKMS92, KPP98, LS98, Rev93, Rev97, Sub94]. In this paper we adopt a view that is more general in two ways. Firstly, we study the more general class of *combination* operations. Secondly, the operations are defined in terms of *epistemic states* – structures in the style of Spohn [Spo88, Spo91] – instead of knowledge bases. The combination operations we study are thus closely related to the combination of preferences in the context of group decision making [LL00].

In section 2 we give a brief introduction to the merging of knowledge bases, focussing on the work of Konieczny and Pino-Pérez [KPP98]. This is followed, in section 3, by a description of our framework for the combination of epistemic states. In section 4 we construct a number of combination operations and show how they measure up to proposed properties. Section 5 discusses links between combination operations and the *infobases* of Meyer [Mey99].

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We assume a finitely generated propositional language $L$ closed under the usual propositional connectives and with a classical model-theoretic semantics. $U$ is the set of interpretations of $L$ and $M(\alpha)$ is the set of models of $\alpha \in L$. Classical entailment is denoted by $\models$ and logical equivalence by $\equiv$. We use $\sqcup$ to denote the concatenation of lists. We let $x^n$ denote the list consisting of $n$ versions of $x$. The length of a list $l$ is denoted by $|l|$.

## 2 Merging knowledge bases

In the spirit of the work of Katsuno and Mendelzon [KM91] approaches to the merging of knowledge bases usually represent the beliefs of an agent as a single wff $\phi$ of $L$, known as a *knowledge base*, where $\phi$ represents the set of all wffs entailed by $\phi$. The goal is to construct, from a finite list of such knowledge bases, an appropriate consistent knowledge base in some rational fashion. Konieczny and Pino-Pérez [KPP98] have proposed a general framework for the merging of knowledge bases. A *knowledge list* $e$ is a non-empty finite list of consistent knowledge bases $[\phi_1, \ldots, \phi_{|e|}]$. Two knowledge lists $e_1$ and $e_2$ are *element-equivalent*, written as $e_1 \approx e_2$, iff for every element $\phi_1$ of $e_1$ there is a unique element $\phi_2$ (position-wise) of $e_2$ such that $\phi_1 \equiv \phi_2$ and for every element $\phi_2$ of $e_2$ there is a unique element $\phi_1$ (position-wise) of $e_1$ such that $\phi_2 \equiv \phi_1$. A *KP-merging operation* $\delta$ is a function from the set of all knowledge lists to the set of all knowledge bases satisfying the following postulates (the KP-postulates):

**(KP1)** $\delta(e) \not\models \bot$

**(KP2)** If $\bigwedge_{i=1}^{|e|} \phi_i \not\models \bot$ then $\delta(e) = \bigwedge_{i=1}^{|e|} \phi_i$

**(KP3)** If $e_1 \approx e_2$ then $\delta(e_1) \equiv \delta(e_2)$

**(KP4)** If $\phi_1 \wedge \phi_2 \models \bot$ then $\delta([\phi_1] \sqcup [\phi_2]) \not\models \phi_1$

**(KP5)** $\delta(e_1) \wedge \delta(e_2) \models \delta(e_1 \sqcup e_2)$

**(KP6)** If $\delta(e_1) \wedge \delta(e_2) \not\models \bot$ then $\delta(e_1 \sqcup e_2) \models \delta(e_1) \wedge \delta(e_2)$

(KP1) requires the knowledge base obtained to be satisfiable. (KP2) states that if it is possible to retain all the information contained in a list of knowledge bases (i.e. if they are consistent) then we should do so. (KP3) can be seen as two properties rolled into one. Firstly, it is an appropriate version of Dalal's principle of the irrelevance of syntax [Dal88]: two knowledge lists that are element-equivalent should yield semantically identical results. But the use of element-equivalence also commits us to commutativity. (KP4) is known as the *fairness* postulate; when two knowledge bases are inconsistent it forces us not to prefer one completely over the other. (KP5) and (KP6) correspond to Pareto's conditions in social choice theory [Arr63]. Together they state that whenever the results of merging two knowledge lists are consistent, merging the combined lists should give exactly the same result as retaining all the information when merging the two lists separately.

Konieczny and Pino-Pérez also distinguish between two subclasses of merging operations. An *arbitration* operation tries to take as many differing opinions as possible into account, while the intuition associated with *majority* operations is that the opinion of the majority should prevail. They initially propose the following postulates for arbitration and majority operations.

**(arb)** $\forall n\ \delta(e \sqcup \phi^n) = \delta(e \sqcup [\phi])$

**(maj)** $\exists n\ \delta(e \sqcup \phi^n) \vDash \phi$

It turns out that there is no KP-merging operation satisfying (arb). From this result Konieczny and Pino-Pérez conclude that (arb) is too strong. We are of the opinion that it is not (arb) that is at fault, but some of the KP-postulates. Below we argue against the inclusion of (KP4) as a postulate that needs to be satisfied by all combination operations.

## 3 Combining epistemic states

One of the central points of departure in this paper is that knowledge bases do not have sufficient structure to represent the beliefs of agents in an adequate manner, a point that has been made a number of times in the belief revision literature.[1] As a result, we focus on combination operations on the level of *epistemic states*. We see an epistemic state as providing a plausibility ranking of the interpretations of $L$; the lower the number assigned to an interpretation, the more plausible it is deemed to be.

**Definition 3.1** An epistemic state $\Phi$ is a function from $U$ to the set of natural numbers. Given an epistemic state $\Phi$, the knowledge base associated with $\Phi$, denoted by $\phi_\Phi$, is some $\phi \in L$ such that $M(\phi) = \{u \mid \Phi(u) = 0\}$. ■

This representation of an epistemic state and its associated knowledge base can be traced back to the work of Spohn [Spo88, Spo91]. An epistemic state contains, not just the current beliefs of an agent (in the form of a knowledge base), but also information about the importance, or entrenchment, that an agent attaches to specific beliefs; information that is used to guide the combination process. As an extreme example, an epistemic state with an inconsistent associated knowledge base still contains useful information since it is able to differentiate between different levels of plausibility, or entrenchment, of beliefs.

### 3.1 Preliminaries

An *epistemic list* $E = [\Phi^E_1, \ldots, \Phi^E_{|E|}]$ is a finite list of epistemic states. It is instructive to view an epistemic list pictorially as in figure 1. While such a pictorial view is only useful in representing epistemic lists containing two elements,

[1] See, for example, Darwiche and Pearl [DP97].

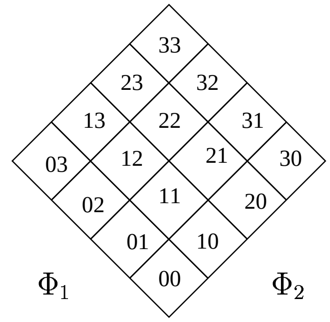


Figure 1: A pictorial representation of an epistemic list containing two epistemic states $\Phi_1$ and $\Phi_2$. The sequence of two digits in each cell above indicates the natural numbers associated with interpretations by the two epistemic states. A cell containing the sequence $ij$ indicates the placement of those interpretations assigned the value $i$ by $\Phi_1$ and assigned the value $j$ by $\Phi_2$.

it serves as a good foundation for understanding the principles underlying the combination of epistemic states in general. For any epistemic state $\Phi$, let

$$\min(\Phi) = \min\{\Phi(u) \mid u \in U\},$$

let

$$\max(\Phi) = \max\{\Phi(u) \mid u \in U\},$$

and for an epistemic list $E$, let

$$\max(E) = \max\{\max(\Phi_i^E) \mid 1 \leq i \leq |E|\}.$$

For an epistemic list $E$ and $u \in U$, let $\min^E(u) = \min\{\Phi_i^E(u) \mid 1 \leq i \leq |E|\}$ and let $\max^E(u) = \max\{\Phi_i^E(u) \mid 1 \leq i \leq |E|\}$. We denote by $seq(E)$ the set of all sequences of length $|E|$ of natural numbers, ranging from 0 to $\max(E)$. We denote by $seq_{\leq}(E)$ the subset of $seq(E)$ of all sequences that are ordered non-decreasingly, and by $seq_{\geq}(E)$ the subset of $seq(E)$ of all sequences that are ordered non-increasingly. For $u \in U$, we let $s^E(u)$ be the sequence containing the natural numbers $\Phi_1^E(u), \ldots, \Phi_{|E|}^E(u)$ in that order, we let $s_{\leq}^E(u)$ be the sequence $s^E(u)$ ordered non-decreasingly, and we let $s_{\geq}^E(u)$ be the sequence $s^E(u)$ ordered non-increasingly. Clearly $s^E(u) \in seq(E)$, $s_{\leq}^E(u) \in seq_{\leq}(E)$ and $s_{\geq}^E(u) \in seq_{\geq}(E)$. Furthermore, we denote by $s_i^E(u)$, $s_i^{(E,\leq)}$ and $s^{(E,\geq)}$ respectively the $i$-th digit in $s^E(u)$, $s_{\leq}^E(u)$ and $s_{\geq}^E(u)$ respectively. Given any set $seq$ of finite sequences of natural numbers and a total preorder $\sqsubseteq$ on $seq$, we define the function $\Omega_{\sqsubseteq}^{seq} : seq \rightarrow \{0, \ldots, |seq| - 1\}$ by assigning consecutive natural numbers to the elements of $seq$ in the order imposed by $\sqsubseteq$, starting by assigning 0 to the elements lowest down in $\sqsubseteq$. We denote the *lexicographic* ordering on $seq$ by $\sqsubseteq_{lex}$.

### 3.2 Properties for combining epistemic states

A *combination operation* $\Delta$ *on epistemic states* is a function from the set of all non-empty epistemic lists to the set of all epistemic states. We propose the following basic properties for the combination of epistemic states:

**(E0)** $\Delta([\Phi])(u) = \Phi(u) - \min(\Phi)$

**(E1)** $\exists u$ s.t. $\Delta(E)(u) = 0$

**(E2)** $\Phi_i^E(u) = \Phi_j^E(u)\ \forall i, j \in \{1, \ldots, |E|\}$ and $s_{\leq}^E(u) \sqsubset_{lex} s_{\leq}^E(v)$ implies that $\Delta(E)(u) < \Delta(E)(v)$

**(E3)** If $\Phi_i^E(u) \leq \Phi_i^E(v)\ \forall i \in \{1, \ldots, |E|\}$ then $\Delta(E)(u) \leq \Delta(E)(v)$

**(E4)** If $\Delta(E)(u) \leq \Delta(E)(v)$ then $\Phi_i^E(u) \leq \Phi_i^E(v)$ for some $i \in \{1, \ldots, |E|\}$

(E0) is the reasonable requirement that trivial combination (with a singleton list) leaves matters unchanged, except in the case where the associated knowledge base is inconsistent. (E1) is simply a restatement of (KP1) in this more general framework. (E2) is a generalisation of (KP2). This can be explained as follows. A knowledge base $\phi$ is a crude epistemic state in the sense that the models of $\phi$ are deemed to be strictly more plausible than its countermodels. So, semantically speaking, (KP2) states that those interpretations that are considered to be most plausible by all of the knowledge bases in a knowledge list are strictly more plausible than any of the remaining interpretations. In this case the "remaining interpretations " can be decribed as those regarded by every knowledge base to be at most as plausible as the most plausible ones, but less plausible by at least one of the knowledge bases. In the same vein (E2) says that whenever all of the epistemic states in $E$ agree on the level of plausibility of a particular interpretation $u$ then, in the epistemic state obtained from a combination operation, $u$ should be strictly more plausible than any interpretation $v$ which is regarded by every epistemic state to be at most as plausible as $u$, but less plausible than $u$ by at least one of the epistemic states. (E3) states that if all epistemic states in $E$ agree that $u$ is at least as plausible as $v$, then so should the resulting epistemic state. (E4) expects justification for regarding an interpretation $u$ as at least as plausible as $v$ after combination has taken place: there has to be at least one epistemic state in $E$ which regards $u$ as at least as plausible as $v$. (E4) is a restatement of the Pareto Principle (in its contrapositive form), one of the properties used to establish Arrow's impossibility theorem in social choice theory [Arr63].

The following fundamental principle for the combination of epistemic states follows easily from (E3):

**(Unit)** If $\Phi_i^E(u) = \Phi_i^E(v)\ \forall i \in \{1, \ldots, |E|\}$ then $\Delta(E)(u) = \Delta(E)(v)$

(Unit) requires interpretations that are treated identically by all epistemic states in an epistemic list to be treated identically in the epistemic state resulting from a combination operation.[2]

Two epistemic lists $E_1$ and $E_2$ are *element-equivalent*, written as $E_1 \approx E_2$, iff for every element $\Phi_1$ of $E_1$ there is a unique element $\Phi_2$ (position-wise) of $E_2$ such that $\Phi_1 = \Phi_2$ and for every element $\Phi_2$ of $E_2$ there is a unique element $\Phi_1$ (position-wise) of $E_1$ such that $\Phi_2 = \Phi_1$. The following property is a generalisation of (KP3). It is instructive to note that the definition of epistemic states obviates the need for any principle of irrelevance of syntax. (Comm) is thus simply a commitment to commutativity.

**(Comm)** $E_1 \approx E_2$ implies $\Delta(E_1) = \Delta(E_2)$

We do not think that (Comm) is an appropriate property for all combination operations. Instead, (Comm) should be seen as a postulate picking out an interesting subclass of combination operations which includes the *merging* operations on epistemic states. By way of justifying this claim we take a brief look in section 4.5 at a class of operations that seem to be valid combination operations, but that do not satisfy (Comm).

Let $\mathcal{E}$ be a finite list of epistemic lists $\mathcal{E} = [E_1, \ldots, E_{|\mathcal{E}|}]$. We also consider the following two properties:

**(E5)** $\Delta(E_i)(u) \leq \Delta(E_i)(v)$ $\forall i \in \{1, \ldots, |\mathcal{E}|\}$ implies that $\Delta(\bigsqcup_{i=1}^{|\mathcal{E}|} E_i)(u) \leq \Delta(\bigsqcup_{i=1}^{|\mathcal{E}|} E_i)(v)$

**(E6)** If $\Delta(\bigsqcup_{i=1}^{|\mathcal{E}|} E_i)(u) \leq \Delta(\bigsqcup_{i=1}^{|\mathcal{E}|} E_i)(v)$ then $\Delta(E_i)(u) \leq \Delta(E_i)(v)$ for some $i \in \{1, \ldots, |\mathcal{E}|\}$

In the presence of (E0), and with the restriction to epistemic states of consistent associated knowledge bases, it is clear that (E5) generalises (E3) and (E6) generalises (E4). In fact, in the presence of (E1), (E5) also implies (KP5).

**Proposition 3.2** *Let $\Delta$ be a combination operation on epistemic states that satisfies (E1) and (E5). Then $\Delta$ also satisfies (KP5).*

**Proof:** Pick any $u$ such that $\Delta(E_1)(u) = \Delta(E_2)(u) = 0$ (if such a $u$ does not exist, the result is trivial). We have to show that $\Delta(E_1 \sqcup E_2)(u) = 0$. It follows from (E5) that $\Delta(E_1 \sqcup E_2)(u) \leq \Delta(E_1 \sqcup E_2)(v)$ for all $v \in U$. And from (E1) it follows that $\Delta(E_1 \sqcup E_2)(u) = 0$ ■

The arbitration postulate (arb) and the majority postulate (maj) can be generalised as follows for combination operations on epistemic states:

**(Arb)** $\forall n$ $\Delta(E \sqcup [\Phi])(u) \leq \Delta(E \sqcup [\Phi])(v)$ iff $\Delta(E \sqcup \Phi^n)(u) \leq \Delta(E \sqcup \Phi^n)(v)$

**(Maj)** $\exists n$ s.t. $\forall u, v \in U$, $\Phi(u) \leq \Phi(v)$ if $\Delta(E \sqcup \Phi^n)(u) \leq \Delta(E \sqcup \Phi^n)(v)$

[2]Note that (Unit) does not imply (E0).

(Arb) is a straightforward generalisation of (arb), but perhaps (Maj) needs some explanation. Semantically speaking (maj) says that the addition of enough instances of the knowledge base $\phi$ to the knowledge list $e$ ensures that the models of the result obtained when merging is a subset of the models of $\phi$. In other words, merging *refines* the knowledge base $\phi$. Similarly, (Maj) states that the addition of enough instances of the epistemic state $\Phi$ to the epistemic list $E$ ensure that the result obtained when combining epistemic states is a refined version of $\Phi$.

It is easily established that, in the presence of (Comm), (Arb) and (Maj) cannot both be satisfied.[3]

**Proposition 3.3** *If $\Delta$ satisfied (Comm) then it cannot satisfy both (Maj) and (Arb).*

**Proof:** Assume that $\Delta$ satisfies (Comm), (Arb) and (Maj). From (Arb) and (Maj) it follows that $\Phi(u) \leq \Phi(v)$ if $\Delta(E \sqcup [\Phi])(u) \leq \Delta(E \sqcup [\Phi])(v)$. Now pick a $u$ and a $v$, and a $\Phi_1$ and a $\Phi_2$, such that $\Phi_1(u) < \Phi_1(v)$ and $\Phi_2(v) < \Phi_2(u)$. From the combination of (Arb) and (Maj) it follows that $\Delta([\Phi_2, \Phi_1])(u) < \Delta([\Phi_2, \Phi_1])(v)$, and similarly, that $\Delta([\Phi_1, \Phi_2])(v) < \Delta([\Phi_1, \Phi_2])(u)$. So, by (Comm) we have that $\Delta([\Phi_1, \Phi_2])(u) < \Delta([\Phi_1, \Phi_2])(v)$; a contradiction. ■

The attentive reader will have observed that we do not provide a generalised version of (KP4). This is because we do not regard it as a suitable property for all rational forms of combining epistemic states; not even for the *merging* of epistemic states. Our basic argument is that the models of a knowledge base associated with an epistemic state $\Phi_1$ may sometimes be given such an implausible ranking by an epistemic state $\Phi_2$ that it would seem reasonable to exclude all these models from the models of $\phi_{\Delta([\Phi_1] \sqcup [\Phi_2])}$. As we shall see, *none* of the combination operations we consider in section 4 satisfies (KP4). See section 6 for a more detailed discussion of this property.

Similarly, we have not provided a generalised version of (KP6). It is easily established that (KP6) does not follow from (E6).[4] On the one hand, (KP6) seems to be a reasonable property to require of a combination operation (on knowledge bases). Indeed, being one of the Pareto conditions, it is widely accepted in the social choice theory community as a useful property. However, in section 4 we shall encounter a number of reasonable combination operations which do not satisy (KP6). At present it seems to us that the failure to satisfy (KP6) is due to the shift from knowledge bases to epistemic states, and that a suitably reformulated version of (KP6) will be satisfied by all these operations.

[3] This is a similar to a result by Konieczny and Pino-Pérez [KPP98] on knowledge base merging.

[4] The combination operation $\Delta_{\max}$ defined in section 4.1 satisfies (E0)-(E6) but does not satisfy (KP6).

## 4 Constructing combination operations

In [KPP98] Konieczny and Pino-Pérez discuss several merging operations on knowledge bases using Dalal's measure of distance between interpretations [Dal88]. For any two interpretations $u$ and $v$, let $dist(u,v)$ denote the number of propositional atoms on which $u$ and $v$ differ. The distance $Dist(\phi, u)$ between a knowledge base $\phi$ and an interpretation $u$ is defined as follows: $Dist(\phi, u) = \min\{dist(u, v) \mid v \in M(\phi)\}$. It is clear that this distance measure can be used to define an epistemic state $\Phi$ as follows:

$$\forall u \in U,\ \Phi(u) = Dist(\phi, u).$$

It is easily seen that $\Phi(u) = 0$ iff $u \in M(\phi)$ and therefore $\phi_\Phi \equiv \phi$. Many of the combination operations on epistemic states that we propose below are appropriate generalisations of these merging operations on knowledge bases.

When reading through the remainder of this section, the reader should observe that the construction of every combination operation consists of two steps. In the first step natural numbers are assigned to interpretations. After the completion of this step it will often be the case that *none* of the interpretations have been assigned the value 0. To ensure compliance with (E1) the second step performs an appropriate uniform subtraction of values which we shall refer to as *normalisation*. Much (if not all) of the notation used in the formal construction of the combination operations in this section is defined in section 3.1, and it is suggested that the reader consult this section if any piece of terminology looks unfamiliar.

### 4.1 Arbitration

Inspired by an arbitration operation proposed by Liberatore and Schaerf [LS98] we propose the following combination operation on epistemic states.

**Definition 4.1** If $E$ contains a single epistemic state $\Phi$, let $\Phi^E_{\min} = \Phi$. If not, let $\Phi^E_{\min}(u) = 2\min^E(u)$ if $\Phi^E_i(u) = \Phi^E_j(u)\ \forall i, j \in \{1, \ldots, |E|\}$ and $\Phi^E_{\min}(u) = 2\min^E(u) + 1$ otherwise. Then $\Delta_{\min}(E)(u) = \Phi^E_{\min}(u) - \min(\Phi^E_{\min})$. ■

Figure 2 contains a pictorial representation of $\Delta_{\min}$. The construction of $\Delta_{\min}$ can be explained as follows. Identify the interpretations for which there is total agreement among all epistemic states about them being the most plausible, and take these to be the most plausible in the epistemic state resulting from the combination operation. The interpretations on the next level of plausibility is obtained by considering all interpretations which are deemed to be most plausible by at least one epistemic state. For the next level of plausibility we move to the interpretations on which there is total agreement about them being the second most plausible set of interpretations, followed by those interpretations which are regarded as the second most plausible by at least one epistemic state. The process described above is repeated until all levels of plausibility for all the epistemic states have been catered for.

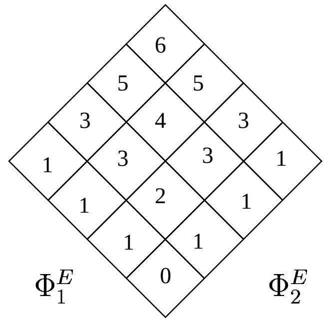


Figure 2: A representation of the combination operation $\Delta_{\min}$. The number in a cell represents the numbers that the appropriate combination operation assigns to the interpretations contained in that cell before normalisation.

**Proposition 4.2** $\Delta_{\min}$ *satisfies (E0)-(E5), (Comm) and (Arb), but it does not satisfy (KP6), (E6) or (Maj). It satisfies (KP4) only if the knowledge bases associated with epistemic states are consistent.*

**Proof:** The satisfaction of (E0) and (E1) are trivial. For (E2), pick a $u$ such that $\Phi_i^E(u) = \Phi_j^E(u)$ for all $i, j \in \{1, \ldots, |E|\}$. Then $\Phi_{\min}^E(u) = 2\min^E(u)$. Now consider a $v$ such that $s_{\leq}^E(u) \sqsubset_{lex} s_{\leq}^E(v)$. So it has to be the case that $\min^E(u) \leq \min^E(v)$. If $\min^E(u) < \min^E(v)$ then clearly $\Delta_{\min}(E)(u) < \Delta_{\min}(E)(v)$. Otherwise, $\Phi_{\min}^E(v) = 2\min^E(v) + 1$, and so $\Delta_{\min}(E)(u) < \Delta_{\min}(E)(v)$. For (E3), pick a $u$ and a $v$ such that $\Phi_i^E(u) \leq \Phi_i^E(v)$ for all $i \in \{1, \ldots, |E|\}$. Then it has to be the case that $\min^E(u) \leq \min^E(v)$ from which it follows that $\Delta_{\min}(E)(u) \leq \Delta_{\min}(E)(v)$. For (E4) we prove the contrapositive. Pick a $u, v$ such that $\Phi_i^E(v) < \Phi_i^E(u)$ for all $i \in \{1, \ldots, |E|\}$. Then it has to be the case that $\min^E(v) < \min^E(u)$, from which it follows that $\Delta_{\min}(E)(v) < \Delta_{\min}(E)(u)$. For (E5), suppose that $\Delta_{\min}(E_i)(u) \leq \Delta_{\min}(E_i)(v)\ \forall i \in \{1, \ldots, |\mathcal{E}|\}$. Assume that

$$\Delta_{\min}(\bigsqcup_{i=1}^{|\mathcal{E}|} E_i)(v) < \Delta_{\min}(\bigsqcup_{i=1}^{|\mathcal{E}|} E_i)(u).$$

We have two cases. For case 1, suppose that

$$\Delta_{\min}(\bigsqcup_{i=1}^{|\mathcal{E}|} E_i)(v) = 2min^{(\bigsqcup_{i=1}^{|\mathcal{E}|} E_i)}(v).$$

Now it follows that for some $E_i$ such that $\Delta_{\min}(E_i)(v) = \Delta_{\min}(\bigsqcup_{i=1}^{|\mathcal{E}|} E_i)(v)$ (there has to be at least one), $\Delta_{\min}(E_i)(v) < \Delta_{\min}(Ei)(u)$; a contradiction. For case 2, suppose that $\Delta_{\min}(\bigsqcup_{i=1}^{|\mathcal{E}|} E_i)(v) = 2\min^{(\bigsqcup_{i=1}^{|\mathcal{E}|} E_i)}(v) + 1$. Now pick any $E_i$ such that $\min^{E_i}(v) = \min\{\min^{E_j}(v) \mid 1 \leq j \leq |\mathcal{E}|\}$ (clearly there is at least one such $E_i$). But then it follows that $\min^{E_i}(v) < \min^{E_i}(u)$, and therefore $\Delta^{E_i}_{\min}(v) < \Delta^{Ei}_{\min}(u)$; a contradiction. The proofs that (Comm) and (Arb) hold are trivial. From proposition 3.3 it then follows that $\Delta_{\min}$ does not satisfy (Maj).

For a counterexample to (KP6), consider three interpretations $u$, $v$ and $w$, and three epistemic states $\Phi_1$, $\Phi_2$ and $\Phi_3$ such that $\Phi_1(u) = \Phi_2(w) = \Phi_3(u) = \Phi_3(v) = \Phi_3(w) = 0$, $\Phi_1(v) = \Phi_1(w) = \Phi_2(u) = \Phi_2(v) = 1$, and let $\Phi_1(x) = \Phi_2(x) = \Phi_3(x) = 2$ for all other interpretations. Now, let $E_1 = [\Phi_1, \Phi_2]$ and let $E_2 = [\Phi_3]$. It is easily seen that the knowledge bases associated with $\Delta_{\min}(E_1)$ and $\Delta_{\min}(E_2)$ are consistent, and that $\Delta_{\min}(E_1 \sqcup E_2)(v) = \Delta_{\min}(E_2)(v) = 0$, but that $\Delta_{\min}(E_1)(v) = 1$. For a counterexample to (E6), consider two interpretations $u$ and $v$, and three epistemic states $\Phi_4$, $\Phi_5$ and $\Phi_6$ such that $\Phi_4(v) = \Phi_5(u) = \Phi_5(v) = 0$, $\Phi_4(u) = \Phi_6(v) = 1$, $\Phi_6(u) = 2$, and let $\Phi_4(x) = \Phi_5(x) = \Phi_6(x) = 3$ for all other interpretations. Now, let $E_3 = [\Phi_4]$, $E_4 = [\Phi_4, \Phi_5]$ and $E_5 = [\Phi_6]$. It can be verified that $\Delta_{\min}(E_3 \sqcup E_4 \sqcup E_5)(u) \leq \Delta_{\min}(E_3 \sqcup E_4 \sqcup E_5)(v)$, but that $\Delta_{\min}(E_i(v) < \Delta_{\min}(E_i(u)$ for $i = 3, 4, 5$.

For (KP4), suppose that $\{w \mid \Phi_1(w) = 0\} \neq \emptyset$, $\{w \mid \Phi_2(w) = 0\} \neq \emptyset$, but that $\{w \mid \Phi_1(w) = 0\} \cap \{w \mid \Phi_2(w) = 0\} = \emptyset$. Then $\{w \mid \Delta_{\min}([\Phi_1, \Phi_2])(w) = 0\} = \{w \mid \Phi_1(w) = 0\} \cup \{w \mid \Phi_2(w) = 0\}$, from which the result follows easily. To show that (KP4) is not satisfied if the associated knowledges base may be inconsistent, simply choose $\Phi_1$ so that its associated knowledge base is inconsistent and choose $\Phi_2$ so that its associated knowledge base is consistent. ■

Next we consider a combination operation that is a generalisation of the $\delta_{\max}$ operation of Konieczny and Pino-Pérez, which was inspired by an example of Revesz's model-fitting operations [Rev97].

**Definition 4.3** Let $\Phi^E_{\max}(u) = \max^E(u)$. Then $\Delta_{\max}(E)(u) = \Phi^E_{\max}(u) - \min(\Phi^E_{\max})$. ■

Figure 3 contains a pictorial representation of $\Delta_{\max}$. It assigns levels of plausibility by looking at the maximum level of plausibility assigned to an interpretation by any of the epistemic states.

**Proposition 4.4** $\Delta_{\max}$ *satisfies (E0)-(E6), (Comm) and (Arb), but it does not satisfy (KP4), (KP6) or (Maj).*

**Proof:** Consider an epistemic list $E = [\Phi^E_1, \ldots, \Phi^E_n]$. The satisfaction of (E0) and (E1) are trivial. For (E2), pick a $u$ such that $\Phi^E_i(u) = \Phi^E_j(u)$ for all $i, j \in \{1, \ldots, |E|\}$. Now consider a $v$ such that $s^E_{\leq}(u) \sqsubset_{lex} s^E_{\leq}(v)$. Then $\max^E(u) < \max^E(v)$ from which the result follows. For (E3), pick a $u$ and a $v$ such that $\Phi^E_i(u) \leq \Phi^E_i(v)$ for all $i \in \{1, \ldots, |E|\}$. Then it has to be the case that $\max^E(u) \leq \max^E(v)$ from which it follows that $\Delta_{\max}(E)(u) \leq \Delta_{\max}(E)(v)$. For (E4) we prove the contrapositive. Pick a $u, v$ such that $\Phi^E_i(v) < \Phi^E_i(u)$ for all $i \in \{1, \ldots, |E|\}$. Then it has to be the case that

$\max^E(v) < \max^E(u)$, from which it follows that $\Delta_{\max}(E)(v) < \Delta_{\max}(E)(u)$. For (E5), suppose that $\Delta_{\max}(E_i)(u) \leq \Delta_{\max}(E_i)(v)$ for all $i \in \{1, \ldots, |\mathcal{E}|\}$. So $\max^{E_i}(u) \leq \max^{E_i}(v)\ \forall i \in \{1, \ldots, |\mathcal{E}|\}$. Thus $\max\{\max^{E_i}(u) \mid 1 \leq i \leq |\mathcal{E}|\} \leq \max\{\max^{E_i}(v) \mid 1 \leq i \leq |\mathcal{E}|\}$. And therefore $\max^{\sqcup_{i=1}^{i=|\mathcal{E}|} E_i}(u) \leq \max^{\sqcup_{i=1}^{i=|\mathcal{E}|} E_i}(v)$. That is, $\Delta_{\max}(\sqcup_{i=1}^{i=|\mathcal{E}|} E_i)(u) \leq \Delta_{\max}(\sqcup_{i=1}^{i=|\mathcal{E}|} E_i)(v)$. For (E6) we prove the contrapositive. Pick a $u, v$ such that $\Delta_{\max}(E_i)(v) < \Delta_{\max}(E_i)(u)\ \forall i \in \{1, \ldots, |\mathcal{E}|\}$. So $\max^{E_i}(v) < \max^{E_i}(u)\ \forall i \in \{1, \ldots, |\mathcal{E}|\}$. Thus $\max\{\max^{E_i}(u) \mid 1 \leq i \leq |\mathcal{E}|\} < \max\{\max^{E_i}(v) \mid 1 \leq i \leq |\mathcal{E}|\}$. And therefore $\max^{\sqcup_{i=1}^{i=|\mathcal{E}|} E_i}(u) < \max^{\sqcup_{i=1}^{i=|\mathcal{E}|} E_i}(v)$. That is, $\Delta_{\max}(\sqcup_{i=1}^{i=|\mathcal{E}|} E_i)(u) < \Delta_{\max}(\sqcup_{i=1}^{i=|\mathcal{E}|} E_i)(v)$. The proofs that (Comm) and (Arb) hold are trivial. From proposition 3.3 it then follows that $\Delta_{\min}$ does not satisfy (Maj). For a counterexample to (KP4), let $\Phi_1(u) = \Phi_2(v) = 0$, $\Phi_2(u) = 1$, $\Phi_1(v) = 3$, and $\Phi_1(x) = \Phi_2(x) = 4$ for all remaining interpretations $x$. For a counterexample to (KP6) let $\Phi_1(u) = \Phi_2(v) = \Phi_2(w) = \Phi_3(u) = \Phi_3(w) = 0$, $\Phi_1(v) = \Phi_1(w) = \Phi_3(v) = 1$, $\Phi_2(u) = 2$, and let $\Phi_1(x) = \Phi_2(x) = \Phi_3(x) = 3$ for all other interpretations. Now, let $E_1 = [\Phi_1, \Phi_2]$ and let $E_2 = [\Phi_3]$. ■

Konieczny and Pino-Pérez do not regard $\delta_{\max}$ as a merging operation on knowledge bases since it fails to satisfy (KP6). This is in contrast with our view of $\Delta_{\max}$ as a valid arbitration operation since it satisfies the postulates (E1)-(E6), (Comm) and (Arb).

### 4.2 Consensus

In this section we consider the idea of *consensus* operations in which agreement on the plausibility ranking of interpretations, instead of the ranking itself, is of overriding importance. We start with a very crude version of this idea. In the definition of these operations we use the following notion of *distance*. For

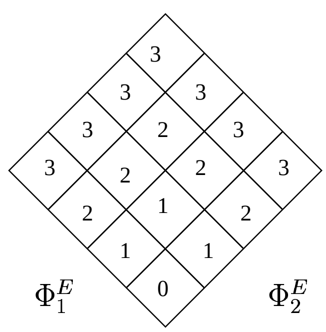


Figure 3: A representation of the combination operation $\Delta_{\max}$. The number in a cell represents the numbers that the appropriate combination operation assigns to the interpretations contained in that cell before normalisation.

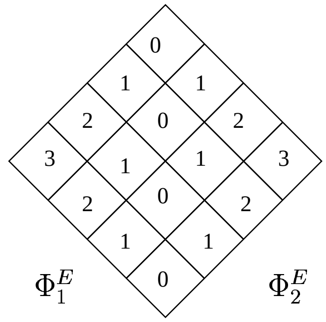


Figure 4: A representation of the combination operation $\Delta_{cons}$. As usual, the number in a cell represents the numbers that the appropriate combination operation assigns to the interpretations contained in that cell before normalisation.

$s \in seq(E)$, let

$$d^E(s) = \sum_{i=1}^{|E|} \sum_{j=i+1}^{|E|} |s_i - s_j|$$

where $s_i$ denotes the $i$th element of $s$.

**Definition 4.5** Define the total preorder $\sqsubseteq$ on $seq(E)$ as follows: $s \sqsubseteq t$ iff $d^E(s) \leq d^E(t)$. Let $\Phi^E_{cons}(u) = \Omega^{seq(E)}_{\sqsubseteq}(s^E(u))$. Then $\Delta_{cons}(E)(u) = \Phi^E_{cons}(u) - \min(\Phi^E_{cons})$. ■

Figure 4 contains a pictorial representation of $\Delta_{cons}$. Informally, $\Delta_{cons}$ considers only the level of agreement of the plausibility ranking of an interpretation and ignores the ranking itself completely. Such an operation seems like a bad idea, and indeed, it fails to satisfy even some of the basic properties.

**Proposition 4.6** $\Delta_{cons}$ *fails to satisfy (E3) and (E4).*

**Proof:** Let $\Phi_1(v) = 0$, $\Phi_2(v) = 1$ and $\Phi_1(u) = \Phi_2(u) = 2$. Then $\Phi_1(v) \leq \Phi_1(u)$ and $\Phi_2(v) \leq \Phi_2(u)$ but $\Delta_{cons}([\Phi_1, \Phi_2])(u) = 0 < \Delta_{cons}([\Phi_1, \Phi_2])(v) = 1$. Also, $\Delta_{cons}([\Phi_1, \Phi_2])(u) = 0 \leq \Delta_{cons}([\Phi_1, \Phi_2])(v) = 1$ but $\Phi_1(v) < \Phi_1(u)$ and $\Phi_2(v) < \Phi_2(u)$. ■

One way to try and force $\Delta_{cons}$ to take account of the plausibility ranking is to refine it follows.

**Definition 4.7** Define the total preorder $\sqsubseteq$ on $seq_{\leq}(E)$ as follows: $s \sqsubseteq t$ iff $d^E(s) < d^E(t)$ or ($d^E(s) = d^E(t)$ and $s \sqsubseteq_{lex} t$). Now, let $\Phi^E_{Rcons}(u) = \Omega^{seq_{\leq}(E)}_{\sqsubseteq}(s^E_{\leq}(u))$. Then $\Delta_{Rcons}(E)(u) = \Phi^E_{Rcons}(u) - \min(\Phi^E_{Rcons})$. ■

Figure 5 contains a pictorial representation of $\Delta_{Rcons}$. $\Delta_{Rcons}$ still regards the agreement on the relative plausibility of interpretations as most important, but

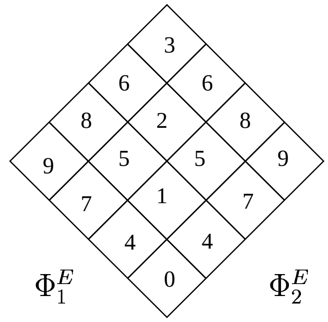


Figure 5: A representation of the combination operation $\Delta_{Rcons}$. As usual, the number in a cell represents the numbers that the appropriate combination operation assigns to the interpretations contained in that cell before normalisation.

if two interpretations are equal in this regard, it uses the plausibility ranking to choose between them.

Unfortunately this refinement is not enough for $\Delta_{Rcons}$ to qualify as a reasonable combination operation.

**Proposition 4.8** $\Delta_{Rcons}$ *fails to satisfy (E3) and (E4).*

**Proof:** The same example can be used as in proposition 4.6. ■

So it seems that consensus is *not* a reasonable way to combine epistemic states. The problem seems to be that consensus operations place too strong an emphasis on agreement and do not take the ranking of interpretations seriously enough. Although neither of the operations defined in this section are reasonable combination operations we shall see in section 4.3 that it may be useful to employ $\Delta_{Rcons}$ to *refine* some combination operations.

### 4.3 Majority

We now turn our attention to majority operations, operations in which the viewpoints of the majority of epistemic states carry the most weight. In our definition of majority operations we shall make use of the following form of *summation.* For $s \in seq(E)$, let

$$sum^E(s) = \sum_{i=1}^{|E|} s_i$$

where $s_i$ is the $i$th element of $s$.

**Definition 4.9** Let $\Phi_\Sigma^E(u) = sum^E(s^E(u))$. Then $\Delta_\Sigma(E)(u) = \Phi_\Sigma^E(u) - \min(\Phi_\Sigma^E)$. ■

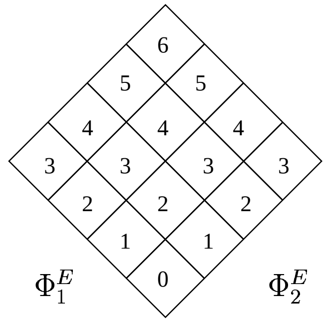


Figure 6: A representation of the combination operation $\Delta_\Sigma$. As usual, the number in a cell represents the numbers that the appropriate combination operation assigns to the interpretations contained in that cell before normalisation.

Figure 6 contains a pictorial representation of $\Delta_\Sigma$. It is an appropriate generalisation of an example by Lin and Mendelzon [LM99] and was also independently proposed by Revesz [Rev93] as an example of weighted model fitting. The idea is simply to obtain the new plausibility ranking of an interpretation by summing the plausibility rankings given by the different epistemic states and then to normalise.

**Proposition 4.10** *$\Delta_\Sigma$ satisfies (E0)-(E6), (Comm), (Maj) and (KP6). It does not satisfy (KP4) or (Arb).*

**Proof:** (E0) and (E1) are trivial and so are (E2), (E3), the contrapositive of (E4), (E5), the contrapositive of (E6) and (Comm). For (Maj), pick any $k$ such that $\Delta_\Sigma(E \sqcup \Phi^k)(u) \leq \Delta_\Sigma(E \sqcup \Phi^k)(v)$ but $\Phi(v) < \Phi(u)$. Now let $l(u,v) = \Delta_\Sigma(E \sqcup \Phi^k)(v) - \Delta_\Sigma(E \sqcup \Phi^k)(u)$. Observe that $\Delta_\Sigma(E \sqcup \Phi^{k+l(u,v)+1})(u) > \Delta_\Sigma(E \sqcup \Phi^{k+l(u,v)+1})(v)$. In this way an $n$ can be found such that $\Delta_\Sigma(E \sqcup \Phi^n)(u) > \Delta_\Sigma(E \sqcup \Phi^n)(v)$ from which the result follows. By proposition 3.3 it also follows that $\Delta_\Sigma$ does not satisfy (Arb). For (KP6), let $\Delta_\Sigma(E_1)(w) = 0$ and $\Delta_\Sigma(E_2)(w) = 0$. Since $\Delta_\Sigma$ satisfies (E1) and (E5) it follows from proposition 3.2 that it also satisfies (KP5), and so $\Delta_\Sigma(E_1 \sqcup E_2)(w) = 0$. Now, pick any $u$ such that $\Delta_\Sigma(E_1 \sqcup E_2)(u) = 0$, and assume that $\Delta_\Sigma(E_1)(u) \neq 0$ or $\Delta_\Sigma(E_2)(u) \neq 0$. Without loss of generality we assume that $\Delta_\Sigma(E_1)(u) \neq 0$. Then $sum^{E_1}(s^{E_1}(w)) < sum^{E_1}(s^{E_1}(u)$. But $sum^{E_1 \sqcup E_2}(u) = sum^{E_1 \sqcup E_2}(w)$ and so it follows that $sum^{E_2}(s^{E_2}(u)) < sum^{E_2}(s^{E_2}(w))$; contradicting the fact that $\Delta_\Sigma(E_2)(w) = 0$. For a counterexample to (KP4) use the counterexample to (KP4) used in the proof of proposition 4.4. ■

The next majority combination operation we consider is an example of how consensus can be usefully employed in the refinement of combination operations. It is a refinement of the majority operation $\Delta_\Sigma$.

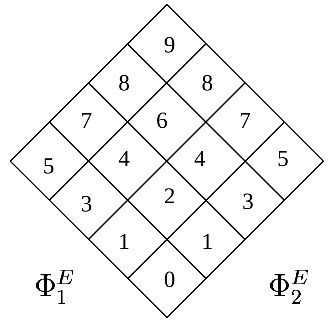


Figure 7: A representation of the combination operation $\Delta_{R\Sigma}$. As usual, the number in a cell represents the numbers that the appropriate combination operation assigns to the interpretations contained in that cell before normalisation.

**Definition 4.11** Define the total preorder $\sqsubseteq$ on $seq(E)$ as follows: $s \sqsubseteq t$ iff $sum^E(s) < sum^E(t)$ or $(sum^E(s) = sum^E(t)$ and $d^E(s) \leq d^E(t))$. Now, let $\Phi^E_{R\Sigma}(u) = \Omega^{seq(E)}_{\sqsubseteq}(s^E(u))$. Then $\Delta_{R\Sigma}(E)(u) = \Phi^E_{R\Sigma}(u) - \min(\Phi^E_{R\Sigma})$. ■

Figure 7 contains a pictorial representation of $\Delta_{R\Sigma}$. $\Delta_{R\Sigma}$ is $\Delta_\Sigma$ refined by using consensus. That is, $\Delta_{R\Sigma}$ determines the plausibility of an interpretation by summing the plausibility level assigned to it by all the epistemic states. And if this results in two interpretations obtaining the same level of plausibility, it tries to distinguish between them further by taking into account the level of agreement about the relative plausibility of each interpretation.

**Proposition 4.12** $\Delta_{R\Sigma}$ *satisfies (E0)-(E4), (Comm) and (Maj). It does not satisfy (KP4), (Arb), (E5)-(E6) and (KP5)-(KP6).*

**Proof:** (E0) is trivial and so are (E1), (E2) and (Comm). For (E3), suppose that $\Phi^E_i(u) \leq \Phi^E_i(v)\ \forall i \in \{1,\ldots,|E|\}$. If $\Phi^E_j(u) < \Phi^E_j(v)$ for some $j \in \{1,\ldots,|E|\}$, it follows that $\Delta_{R\Sigma}(E)(u) \leq \Delta_{R\Sigma}(E)(v)$. Otherwise $d^E(s^E(u)) = d^E(s^E(v))$ from which the result then follows. For the contrapositive of (E4), suppose that $\Phi^E_i(v) < \Phi^E_i(u)\ \forall i \in \{1,\ldots,|E|\}$. Then it has to be the case that $sum^E(v) < sum^E(u)$ and so $\Delta_{R\Sigma}(E)(v) < \Delta_{R\Sigma}(E)(u)$. The proof for (Maj) is identical to the one for (Maj) used in proposition 4.10. For (KP4), take the counterxample to (KP4) used in proposition 4.4. By proposition 3.3 it then also follows that $\Delta_{R\Sigma}$ does not satisfy (Arb). As a counterexample to (E5), let $\Phi_1(u) = \Phi_4(u) = \Phi_4(v) = 0$, $\Phi_1(v) = \Phi_2(v) = \Phi_5(u) = \Phi_5(v) = 1$, $\Phi_2(u) = \Phi_3(u) = 3$, $\Phi_3(v) = 4$, and let $E_1 = [\Phi_1, \Phi_2, \Phi_3]$ and $E_2 = [\Phi_4, \Phi_5]$. As a counterexample to the contrapositive of (E6), let $\Phi_1(v) = 0$, $\Phi_1(u) = \Phi_4(u) = 1$, $\Phi_4(v) = \Phi_5(v) = 2$, $\Phi_2(u) = \Phi_5(u) = 3$, $\Phi_2(v) = \Phi_3(v) = 6$, $\Phi_3(u) = 8$, and let $E_1 = [\Phi_1, \Phi_2, \Phi_3]$ and $E_2 = [\Phi_4, \Phi_5]$. As a counterexample to (KP5), take the counterexample used for (E5) and assume that for any $x$ other than $u$ and $v$, $\Phi_i(x) = 10\ \forall i \in \{1,\ldots,5\}$. Similarly, as a counterexample to (KP6), take the

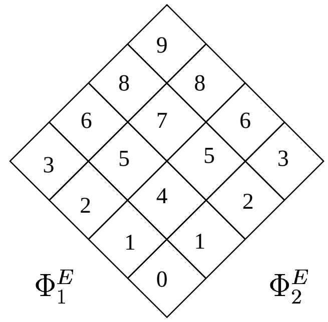


Figure 8: A representation of the combination operation $\Delta_{Rmin}$. The number in a cell represents the numbers that the appropriate combination operation assigns to the interpretations contained in that cell before normalisation.

counterexample for (E6) and for all interpretations $x$ other than $u$ and $v$, let $\Phi_i(x) = 20 \ \forall i \in \{1, \ldots, 5\}$. ■

### 4.4 Other forms of combination

In this section we consider two combination operations that are neither arbitration nor majority operations. The first is a refined version of $\Delta_{\text{min}}$.

**Definition 4.13** Let $\Phi^E_{Rmin}(u) = \Omega^{seq_{\leq}(E)}_{\sqsubseteq_{lex}}(s^E_{\leq}(u))$. Then

$$\Delta_{Rmin}(E)(u) = \Phi^E_{Rmin}(u) - \min(\Phi^E_{Rmin}).$$

■

Figure 8 contains a pictorial representation of $\Delta_{Rmin}$. Informally the construction of $\Delta_{Rmin}$ can be explained as follows. It makes the same relative distinction between the plausibility of interpretations as $\Delta_{\text{min}}$ but it goes further; it distinguishes between interpretations that are regarded as equally plausible by $\Delta_{\text{min}}$ by taking into account all the levels of plausibility assigned to a interpretation by the different epistemic states.

**Proposition 4.14** $\Delta_{Rmin}$ *satisfies (E0)-(E6), (Comm) and (KP6), but it does not satisfy (KP4), (Arb) or (Maj).*

**Proof:** Consider an epistemic list $E = [\Phi^E_1, \ldots, \Phi^E_n]$. The satisfaction of (E0) and (E1) are trivial. (E2) follows immediately from the definition of $\Delta_{Rmin}$. For (E3), suppose that $\Phi^E_i(u) \leq \Phi^E_i(v) \ \forall i \in \{1, \ldots, |E|\}$, and assume that $\Delta_{Rmin}(E)(v) < \Delta_{Rmin}(E)(u)$. Then $\exists i \in \{1, \ldots, |E|\}$ such that $s_i^{(E,\leq)}(v) < s_i^{(E,\leq)}(u)$ and $s_j^{(E,\leq)}(v) = s_j^{(E,\leq)}(u) \ \forall j \in \{1, \ldots, i-1\}$. Since

$s_i^{(E,\leq)}(u) \leq s_j^{(E,\leq)}(u)\ \forall j \in \{i+1,\ldots,|E|\}$ it has to be the case that $s_k^{(E,\leq)}(u) \leq s_i^{(E,\leq)}(v)$ for some $k \in \{1,\ldots,i-1\}$. But this means there has to be an $l \in \{i,\ldots,|E|\}$ and an $m \in \{1,\ldots,i-1\}$ such that $s_l^{(E,\leq)}(u) \leq s_m^{(E,\leq)}(v)$; contradicting the fact that $s_i^{(E,\leq)}(v) < s_i^{(E,\leq)}(u)$. For (E4) we prove the contrapositive. Suppose that $\Phi_i^E(v) < \Phi_i^E(u)\ \forall i \in \{1,\ldots,|E|\}$, and assume that $\Delta_{Rmin}(E)(u) \leq \Delta_{Rmin}(E)(v)$. Then $\exists i \in \{1,\ldots,|E|\}$ such that $s_i^{(E,\leq)}(u) \leq s_i^{(E,\leq)}(v)$ and $s_j^{(E,\leq)}(u) = s_j^{(E,\leq)}(v)\ \forall j \in \{1,\ldots,i-1\}$. Since $s_i^{(E,\leq)}(v) \leq s_j^{(E,\leq)}(v)\ \forall j \in \{i+1,\ldots,|E|\}$ it has to be the case that $s_k^{(E,\leq)}(v) < s_i^{(E,\leq)}(u)$ for some $k \in \{1,\ldots,i-1\}$. But this means there has to be an $l \in \{1,\ldots,i-1\}$ and an $m \in \{i,\ldots,|E|\}$ such that $s_m^{(E,\leq)}(v) < s_l^{(E,\leq)}(u)$, contradicting the fact that $s_i^{(E,\leq)}(u) \leq s_i^{(E,\leq)}(v)$. The proofs for (E5) and (E6) are similar to those for (E3) and (E4) respectively and are omitted, and the proof for (Comm) is trivial. For (KP6) let $w$ be such that $\Delta_{Rmin}(E_1)(w) = 0$ and $\Delta_{Rmin}(E_2)(w) = 0$. Since $\Delta_{Rmin}$ satisfies (E1) and (E5) it follows from proposition 3.2 that it also satisfies (KP5) and so $\Delta_{Rmin}(E_1 \sqcup E_2)(w) = 0$. Now pick any $u$ such that $\Delta_{Rmin}(E_1 \sqcup E_2)(u) = 0$. Then $s_{\leq}^{E_1 \sqcup E_2}(u) = s_{\leq}^{E_1 \sqcup E_2}(w)$. Assume that $\Delta_{Rmin}(E_1)(u) \neq 0$ or $\Delta_{Rmin}(E_2)(u) \neq 0$. Without loss of generality we assume that $\Delta_{Rmin}(E_1)(u) \neq 0$. Then $s_{\leq}^{E_1}(w) \sqsubset_{lex} s_{\leq}^{E_1}(u)$. Now, $s_{\leq}^{E_2}(u) = s_{\leq}^{E_2}(w)$ contradicts $s_{\leq}^{E_1 \sqcup E_2}(u) = s_{\leq}^{E_1 \sqcup E_2}(w)$, and so does $s_{\leq}^{E_2}(w) < s_{\leq}^{E_2}(u)$. So it has to be the case that $s_{\leq}^{E_2}(u) < s_{\leq}^{E_2}(w)$, contradicting the fact that $\Delta_{Rmin}(E_2)(w) = 0$. As a counterexample to (KP4) let $\Phi_1(u) = \Phi_2(v) = 0$, $\Phi_1(v) = 2$, $\Phi_2(u) = 1$, and let $\Phi_1(w) = \Phi_2(w) = 3$ for the remaining interpretations. As a counterexample to (Arb), choose any $E$, $\Phi$, $u$ and $v$ where $\Phi(u) = \Phi_1(v) = 0$, $\Phi(v) = \Phi_1(u) = 2$ and $E = [\Phi_1]$. As a counterexample to (Maj) let $\Phi_1(u) = 0, \Phi_1(v) = 2, \Phi(u) = 4$ and $\Phi(v) = 3$. Then $\Delta_{Rmin}([\Phi_1] \sqcup \Phi^n)(u) \leq \Delta_{Rmin}([\Phi_1] \sqcup \Phi^n)(v)\ \forall n$ even though $\Phi(v) < \Phi(u)$. ■

The next combination operation we consider is a refined version of $\Delta_{\max}$. It can be seen as a generalisation of the $\delta_{Gmax}$ operation of Konieczny and Pino-Pérez which, in turn, was inspired by an example of Revesz's model-fitting operations [Rev97].

**Definition 4.15** Let $\Phi_{Rmax}^E(u) = \Omega_{\sqsubseteq_{lex}}^{seq_{\geq}(E)}(s_{\geq}^E(u))$. Then $\Delta_{Rmax}(E)(u) = \Phi_{Rmax}^E(u) - \min(\Phi_{Rmax}^E)$. ■

Figure 9 is a pictorial representation of $\Delta_{Rmax}$. $\Delta_{Rmax}$ refines $\Delta_{\max}$ in a way that is analogous to the way $\Delta_{Rmin}$ refines $\Delta_{\min}$. Whereas $\Delta_{\max}$ assigns levels of plausibility based only on the maximum level of plausibility assigned to a interpretation by epistemic states, $\Delta_{Rmax}$ also takes into account all the other levels of plausibility assigned to the interpretation.

**Proposition 4.16** *$\Delta_{Rmax}$ satisfies (E0)-(E6), (Comm) and (KP6). It does not satisfy (KP4), (Arb) and (Maj).*

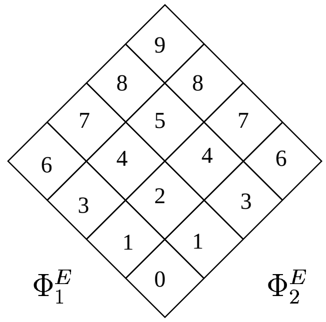


Figure 9: A representation of the combination operation $\Delta_{Rmax}$. The number in a cell represents the numbers that the appropriate combination operation assigns to the interpretations contained in that cell before normalisation.

**Proof:** Consider an epistemic list $E = [\Phi_1^E, \ldots, \Phi_n^E]$. The satisfaction of (E0) and (E1) are trivial. For (E2) observe that if $\Phi_i^E(u) = \Phi_j^E(v)\ \forall i, j \in \{1, \ldots, |E|\}$ and $s_\leq^E(u) \sqsubset_{lex} s_\leq^E(v)$ then $s_\geq^E(u) \sqsubset_{lex} s_\geq^E(v)$, from which the result follows. For (E3), suppose that $\Phi_i^E(u) \leq \Phi_i^E(v)\ \forall i \in \{1, \ldots, |E|\}$, and assume that $\Delta_{Rmax}(E)(v) < \Delta_{Rmax}(E)(u)$. Then $\exists i \in \{1, \ldots, |E|\}$ such that $s_i^{(E,\geq)}(v) < s_i^{(E,\geq)}(u)$ and $s_j^{(E,\geq)}(v) = s_j^{(E,\geq)}(u)\ \forall j \in \{1, \ldots, i-1\}$. Since $s_i^{(E,\geq)}(u) \leq s_j^{(E,\geq)}(u)\ \forall j \in \{1, \ldots, i-1\}$ it has to be the case that $s_k^{(E,\geq)}(u) \leq s_i^{(E,\geq)}(v)$ for some $k \in \{i+1, \ldots |E|\}$. But this means there has to be an $m \in \{i+1, \ldots, |E|\}$ and an $l \in \{1, \ldots, i\}$ such that $s_l^{(E,\geq)}(u) \leq s_m^{(E,\geq)}(v)$; contradicting the fact that $s_i^{(E,\geq)}(v) < s_i^{(E,\geq)}(u)$. For (E4) we prove the contrapositive. Suppose that $\Phi_i^E(v) < \Phi_i^E(u)\ \forall i \in \{1, \ldots, |E|\}$, and assume that $\Delta_{Rmax}(E)(u) \leq \Delta_{Rmax}(E)(v)$. Then $\exists i \in \{1, \ldots, |E|\}$ such that $s_i^{(E,\geq)}(u) \leq s_i^{(E,\geq)}(v)$ and $s_j^{(E,\geq)}(u) = s_j^{(E,\geq)}(v)\ \forall j \in \{1, \ldots, i-1\}$. Since $s_i^{(E,\geq)}(u) \geq s_j^{(E,\geq)}(u)\ \forall j \in \{i+1, \ldots, |E|\}$ it has to be the case that $s_i^{(E,\geq)}(v) < s_k^{(E,\geq)}(u)$ for some $k \in \{1, \ldots, i-1\}$. But this means there has to be an $l \in \{1, \ldots, i-1\}$ and an $m \in \{i, \ldots, |E|\}$ such that $s_l^{(E,\geq)}(v) < s_m^{(E,\geq)}(u)$, contradicting the fact that $s_i^{(E,\geq)}(u) \leq s_i^{(E,\geq)}(v)$. The proofs for (E5) and (E6) are similar to those for (E3) and (E4) respectively and are omitted, and the proof for (Comm) is trivial. For (KP6) let $w$ be such that $\Delta_{Rmax}(E_1)(w) = 0$ and $\Delta_{Rmax}(E_2)(w) = 0$. Since $\Delta_{Rmax}$ satisfies (E1) and (E5) it follows from proposition 3.2 that it also satisfies (KP5) and so $\Delta_{Rmax}(E_1 \sqcup E_2)(w) = 0$. Now pick any $u$ such that $\Delta_{Rmax}(E_1 \sqcup E_2)(u) = 0$. Then $s_\geq^{E_1 \sqcup E_2}(u) = s_\geq^{E_1 \sqcup E_2}(w)$. Assume that $\Delta_{Rmax}(E_1)(u) \neq 0$ or $\Delta_{Rmax}(E_2)(u) \neq 0$. Without loss of generality we assume that $\Delta_{Rmax}(E_1)(u) \neq 0$. Then $s_\geq^{E_1}(w) < s_\geq^{E_1}(u)$. Now, $s_\geq^{E_2}(u) = s_\geq^{E_2}(w)$ contradicts $s_\geq^{E_1 \sqcup E_2}(u) = s_\geq^{E_1 \sqcup E_2}(w)$, and so does $s_\geq^{E_2}(w) < s_\geq^{E_2}(u)$. So it has to be the case that $s_\geq^{E_2}(u) < s_\geq^{E_2}(w)$, contradicting the fact that $\Delta_{Rmax}(E_2)(w) = 0$.

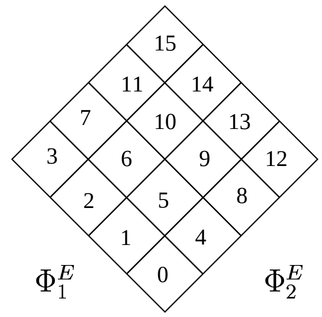


Figure 10: A representation of the combination operation $\Delta_{lex}$. As usual, the number in a cell represents the numbers that the appropriate combination operation assigns to the interpretations contained in that cell before normalisation.

As a counterexample to (KP4), use the counterexample to (KP4) used in the proof of proposition 4.4. For (Arb), let $\Phi_1(v) = \Phi(u) = 0$ and $\Phi_1(u) = \Phi(v) = 2$. Then $\Delta_{Rmax}([\Phi_1] \sqcup [\Phi])(u) = \Delta_{Rmax}([\Phi_1] \sqcup [\Phi])(v)$, but $\Delta_{Rmax}([\Phi_1] \sqcup \Phi^2)(u) < \Delta_{Rmax}([\Phi_1] \sqcup \Phi^2)(v)$. For (Maj), let $\Phi_1(u) = \Phi(v) = 0, \Phi_1(v) = 10$ and $\Phi(u) = 5$. Then $\Delta_{Rmax}([\Phi_1] \sqcup \Phi^n)(u) \leq \Delta_{Rmax}([\Phi_1] \sqcup \Phi^n)(v)\ \forall n$, but $\Phi(v) < \Phi(u)$. ■

The fact that we do not regard $\Delta_{Rmax}$ as an arbitration operation is in conflict with the view of Konieczny and Pino-Pérez who regard $\delta_{Gmax}$ as an arbitration operation on knowledge bases even though the latter does not satisfy (arb).[5]

### 4.5 Non-commutative combination

Thus far we have restricted ourselves to the construction of *commutative* combination operations – satisfying (Comm) – but a complete description of combination operations ought to take into account constructions such as that of Nayak [Nay94], in which the combination of two epistemic states is obtained by a lexicographic refinement of one by the other. We present here a generalised version of Nayak's proposal. For this case the epistemic states in an epistemic list are assumed to be ranked according to reliability. That is, given an epistemic list $E = [\Phi_1^E, \ldots, \Phi_{|E|}^E]$, $\Phi_i^E$ is at least as reliable as $\Phi_j^E$ iff $i \leq j$.

**Definition 4.17** Let $\Phi_{lex}^E(u) = \Omega_{\sqsubseteq_{lex}}^{seq(E)}(s^E(u))$. Then $\Delta_{lex}(E)(u) = \Phi_{lex}^E(u) - \min(\Phi_{lex}^E)$. ■

Figure 10 contains a pictorial representation of $\Delta_{lex}$. It satisfies all the basic properties for combination operations.

[5] It satisfies their weaker version of (arb).

**Proposition 4.18** $\Delta_{lex}$ *satisfies (E0)-(E6), as well as (KP5)-(KP6). It does not satisfy (KP4) and (Comm).*

**Proof:** (E0) and (E1) are trivial. For (E2) suppose that $\Phi_i(u) = \Phi_j(u)\ \forall i, j \in \{1, \ldots, |E|\}$, and let $s^E_{\leq}(u) \sqsubset_{lex} s^E_{\leq}(v)$. This means $s^E(u) \sqsubset_{lex} s^E(v)$ from which the result follows. For (E3), suppose that $\Phi_i(u) \leq \Phi_i(v)\ \forall i \in \{1, \ldots, |E|\}$. Then $s^E(u) \sqsubseteq_{lex} s^E(v)$, from which the result follows. For the contrapositive of (E4), suppose that $\Phi_i(v) < \Phi_i(u)\ \forall i \in \{1, \ldots, |E|\}$. Then $s^E(v) \sqsubset_{lex} s^E(u)$, from which the result follows. For (E5), suppose that $\Delta_{lex}(E_i)(u) \leq \Delta_{lex}(E_i)(v)\ \forall i \in \{1, \ldots, |\mathcal{E}|\}$. That is, $s^{E_i}(u) \sqsubseteq_{lex} s^{E_i}(v)\ \forall i \in \{1, \ldots, |\mathcal{E}|\}$. But then $s^{\sqcup_{i=1}^{|\mathcal{E}|} E_i}(u) \sqsubseteq_{lex} s^{\sqcup_{i=1}^{|\mathcal{E}|} E_i}(v)$ from which the result follows. For the contrapositive of (E6), suppose that $\Delta_{lex}(E_i)(v) < \Delta_{lex}(E_i)(u)\ \forall i \in \{1, \ldots, |\mathcal{E}|\}$. That is, $s^{E_i}(v) \sqsubset_{lex} s^{E_i}(u)\ \forall i \in \{1, \ldots, |\mathcal{E}|\}$. But then it is the case that $s^{\sqcup_{i=1}^{|\mathcal{E}|} E_i}(u) \sqsubset_{lex} s^{\sqcup_{i=1}^{|\mathcal{E}|} E_i}(v)$ from which the result follows. The failure of (KP4) is trivial. To see that (Comm) isn't satisfied, let $\Phi_1(u) = \Phi_2(v) = 0, \Phi_1(v) = \Phi_2(u) = 1$, and let $\Phi_1(x) = \Phi_2(x) = 2$ for all other interpretations $x$. Then $[\Phi_1, \Phi_2] \approx [\Phi_2, \Phi_1]$ but $\Delta_{lex}([\Phi_1, \Phi_2]) \neq \Delta_{lex}([\Phi_2, \Phi_1])$. ■

It is easily shown that $\Delta_{lex}$ does not satisfy (Maj). Also, the fact that $\Delta_{lex}$ is not commutative can be exploited to phrase (Arb) in such a way that $\Delta_{lex}$ fails to satisfy it.

**(Arb')** $\forall n\ \Delta(E \sqcup [\Phi])(u) \leq \Delta(E \sqcup [\Phi])(v)$ iff $\Delta(\Phi^n \sqcup E)(u) \leq \Delta(\Phi^n \sqcup E)(v)$

**Proposition 4.19** *If* $\Delta$ *satisfies (Comm) then it satisfies (Arb) iff it satisfies (Arb').*

**Proof:** The proof is trivial and is omitted. ■

**Proposition 4.20** $\Delta_{lex}$ *does not satisfy (Arb') and (Maj).*

**Proof:** Let $\Phi_1(u) = \Phi_2(v) = 0$ and $\Phi_1(v) = \Phi_2(u) = 1$. For (Arb'), observe that $\Delta_{lex}([\Phi_1, \Phi_2])(u) < \Delta_{lex}([\Phi_1, \Phi_2])(v)$, but $\Delta_{lex}([\Phi_2, \Phi_1, \Phi_2])(v) < \Delta_{lex}([\Phi_2, \Phi_1, \Phi_2])(u)$. For (Maj) observe that $\Delta_{lex}([\Phi_1] \sqcup \Phi_2^n)(u) \leq \Delta_{lex}([\Phi_1] \sqcup \Phi_2^n)(v)\ \forall n$ but that $\Phi_2(v) < \Phi_2(u)$. ■

So $\Delta_{lex}$ seems to be a reasonable combination operation although it is neither an arbitration nor a majority operation and is not commutative.

## 5 Combination and infobases

Our description of combination operations uses a representation of epistemic states as functions assigning a plausibility ranking to the interpretations of $L$, but where do these plausibility rankings come from? One way in which to generate them is by using the *infobases* of Meyer [Mey99]. An infobase is a finite list of wffs. Intuitively it is a structured representation of the beliefs of an agent with a foundational flavour. It is assumed that every wff in an infobase is obtained independently. Meyer uses an infobase to define a total preorder on $U$, which is then used to perform belief change. However, we can also use an infobase to define an epistemic state. The idea is to consider the number of times that an interpretation occurs as a model of one of the wffs in an infobase: the more it occurs, the higher its plausibility ranking.

**Definition 5.1** For $u \in U$, define the *IB*-number $u_{IB}$ of $u$ as the number of elements $\alpha$ in an infobase *IB* such that $u \in M(\alpha)$, and let

$$\max(IB) = \max\{u_{IB} \mid u \in U\}.$$

Now we define the epistemic state $\Phi^{IB}$ associated with *IB* as follows: for $u \in U$, $\Phi^{IB}(u) = \max(IB) - u_{IB}$.[6] ■

Observe that an inconsistent infobase still contains useful information about an agent's beliefs. In fact, the knowledge base associated with an epistemic state $\Phi^{IB}$ is always consistent, regardless of whether the wffs in *IB* are jointly consistent.

Below we show that, under certain circumstances, infobases seem to provide a natural setting in which to apply combination operations. This is not a detailed investigation, but should simply seen as providing corroborating evidence for the claim that there are useful links to be explored between infobases and combining epistemic states.

Firstly, define an *infobase list* $EB = [IB_1, \ldots, IB_{|EB|}]$ as a finite non-empty list of infobases and let $E^{EB}$ denote the epistemic list $[\Phi^{IB_1}, \ldots, \Phi^{IB_{|EB|}}]$ of epistemic states associated with the infobases occurring in *EB*. It turns out that concatenating the different infobase lists into one big infobase list yields exactly the same associated epistemic state as the one obtained when $\Delta_\Sigma$ is applied to $E^{EB}$.

**Proposition 5.2** *Consider the infobase list* $EB = [IB_1, \ldots, IB_{|EB|}]$ *and let* $IB = \bigsqcup_{i=1}^{|EB|} IB_i$. *Then* $\Delta_\Sigma(E^{EB}) = \Phi^{IB}$.

[6] It is easily verified that adding tautologies to an infobase *IB* does not alter the epistemic state associated with it.

**Proof:**

$$
\begin{aligned}
& \Delta_\Sigma(E^{EB})(u) \\
= \; & \Sigma_{i=1}^{|EB|}\Phi^{IB_i}(u) - \min(\Phi_\Sigma^{E^{EB}}) \\
= \; & \Sigma_{i=1}^{|EB|}(\max\{v_{IB_i} \mid v \in U\} - u_{IB_i}) - \min\{\Phi_\Sigma^{E^{EB}}(w) \mid w \in U\} \\
= \; & \Sigma_{i=1}^{|EB|}(\max\{v_{IB_i} \mid v \in U\} - u_{IB_i}) \\
& - \min\{\Sigma_{i=1}^{|EB|}(\max\{v_{IB_i} \mid v \in U\} - w_{IB_i}) \mid w \in U\} \\
= \; & \Sigma_{i=1}^{|EB|}\max\{v_{IB_i} \mid v \in U\} - \Sigma_{i=1}^{|EB|}u_{IB_i} \\
& -\Sigma_{i=1}^{|EB|}\max\{v_{IB_i} \mid v \in U\} + \max\{\Sigma_{i=1}^{|EB|}w_{IB_i} \mid w \in U\} \\
= \; & \max\{\Sigma_{i=1}^{|EB|}w_{IB_i} \mid w \in U\} - \Sigma_{i=1}^{|EB|}u_{IB} \\
= \; & \max\{w_{IB} \mid w \in U\} - u_{IB} \\
= \; & \Phi^{IB}(u)
\end{aligned}
$$

■

Secondly, Konieczny and Pino-Pérez [KPP98] give a convincing example to show that we may sometimes want to include, as models of $\delta(e)$, interpretations other than the models of the knowledge bases in $e$. Below is a scaled down version of their example.

**Example 5.3** We want to speculate on the stock exchange and we ask two equally reliable financial experts about two shares. Let the atom $p$ denote the fact that share 1 will rise and $q$ the fact that share 2 will rise. The first expert says that both shares will rise: $\phi_1 = p \wedge q$, while the second one believes that both shares will fall: $\phi_2 = \neg p \wedge \neg q$. Intuitively it seems reasonable to conclude that both experts are right (and wrong) about exactly one share, although we don't know which share in either case. That is, we require the result of combining these two knowledge bases to be such that $M(\delta([\phi_1] \sqcup [\phi_2])) = \{10, 01\}$.[7] Observe that $M(\delta([\phi_1] \sqcup [\phi_2])) \not\subseteq M(\phi_1) \cup M(\phi_2)$. ■

An analysis of this example shows that both experts are assumed to make an implicit assumption of independence of the performance of the shares. In other words, both see the performance of one share as completely independent of the other (and vice versa). And it is precisely this kind of independence that can be captured by the use of infobases. The beliefs of the first expert can thus be expressed most accurately as the infobase $IB_1 = [p, q]$ and the beliefs of the second expert as the infobase $IB_2 = [\neg p, \neg q]$. The epistemic states obtained from these two infobases are: $\Phi^{IB_1}(11) = 0, \Phi^{IB_1}(10) = \Phi^{IB_1}(01) = 1, \Phi^{IB_1}(00) = 2$, and $\Phi^{IB_2}(00) = 0, \Phi^{IB_2}(10) = \Phi^{IB_2}(01) = 1, \Phi^{IB_2}(11) = 2$. It can be verified that $\Delta_{\max}(E^{EB}) = \Delta_{Rmax}(E^{EB}) = \Delta_{R\Sigma}(E^{EB}) = \Phi$, where $EB = [IB_1, IB_2]$, $\Phi(10) = \Phi(01) = 0$ and $\Phi(11) = \Phi(00) = 1$. So $\Delta_{R\Sigma}$, $\Delta_{\max}$ and $\Delta_{Rmax}$ yield the results corresponding to our intuition for this example.

[7] We represent interpretations as sequences consisting of 0s (representing falsity) and 1s (representing truth), where the first digit in a sequence represents the truth value of $p$ and the second one the truth value of $q$.

| Properties | Combination operations | | | | | | |
|---|---|---|---|---|---|---|---|
| | $\Delta_{\min}$ | $\Delta_{\max}$ | $\Delta_{\Sigma}$ | $\Delta_{R\Sigma}$ | $\Delta_{Rmin}$ | $\Delta_{Rmax}$ | $\Delta_{lex}$ |
| (E0) | $\surd$ | $\surd$ | $\surd$ | $\surd$ | $\surd$ | $\surd$ | $\surd$ |
| (E1) | $\surd$ | $\surd$ | $\surd$ | $\surd$ | $\surd$ | $\surd$ | $\surd$ |
| (E2) | $\surd$ | $\surd$ | $\surd$ | $\surd$ | $\surd$ | $\surd$ | $\surd$ |
| (E3) | $\surd$ | $\surd$ | $\surd$ | $\surd$ | $\surd$ | $\surd$ | $\surd$ |
| (E4) | $\surd$ | $\surd$ | $\surd$ | $\surd$ | $\surd$ | $\surd$ | $\surd$ |
| (E5) | $\surd$ | $\surd$ | $\surd$ | x | $\surd$ | $\surd$ | $\surd$ |
| (E6) | x | $\surd$ | $\surd$ | x | $\surd$ | $\surd$ | $\surd$ |
| (KP4) | x | x | x | x | x | x | x |
| (KP5) | $\surd$ | $\surd$ | $\surd$ | x | $\surd$ | $\surd$ | $\surd$ |
| (KP6) | x | x | $\surd$ | x | $\surd$ | $\surd$ | $\surd$ |
| (Arb) | $\surd$ | $\surd$ | x | x | x | x | - |
| (Arb') | $\surd$ | $\surd$ | x | x | x | x | x |
| (Maj) | x | x | $\surd$ | $\surd$ | x | x | x |
| (Comm) | $\surd$ | $\surd$ | $\surd$ | $\surd$ | $\surd$ | $\surd$ | x |

Table 1: A summary of the combination operations considered in this paper and the properties they satisfy. The table does not contain $\Delta_{cons}$ and $\Delta_{Rcons}$ which were discussed in section 4.2.

## 6 Conclusion

Table 6 contains all the combination operations we have constructed, except for the two consensus operations discussed in section 4.2, and indicates which properties they satisfy. These results are consistent with the view that (E0)-(E4) may be regarded as basic postulates for combination operations on epistemic states. The status of (E5) and (E6) is less clear. The main obstacle to satisfying these properties seems to be the normalisation process to ensure that at least one interpretation has a plausibility ranking of 0.

From the results obtained it also seems clear that (KP4) is too strong a property to insist on. It is interesting to observe that the constructions of Konieczny and Pino-Pérez [KPP98], on which some of our constructions are based, all satisfy (KP4), while none of ours do. This apparent anomalous behaviour can be explained by noting that the Konieczny and Pino-Pérez constructions are based on a measure of distance between interpretations which is more restrictive than the notion of distance used in epistemic states.

This brings us to the question of whether (KP6) is a reasonable property to impose on all combination operations. Informally, (KP6) tries to bring in a kind of independence by insisting that the concatenation of two epistemic lists be at least as strong, logically speaking, as conjunction. Together with (KP5), (KP6) states that if the combination of $E_1$ is consistent with the combination of $E_2$ then the combination of $E_1 \sqcup E_2$ should yield the same knowledge base as the conjunction of the separate combination process. On the one hand then, (KP6) seems like a reasonable property to impose on all combination operations. Unfortunately some of the reasonable combination properties, $\Delta_{\max}$, $\Delta_{\min}$ and $\Delta_{R\Sigma}$, do not satisfy (KP6). At present we suspect that it is the move from knowledge bases to epistemic states which causes this anomalous behaviour, and that a version of (KP6), appropriately modified for epistemic states will indeed by satisfied by these operations.

Our results suggest that (Arb) is an appropriate postulate for the subclass of arbitration operations. This position conflicts, to some extent, with that of Konieczny and Pino-Pérez [KPP98] who argue against the use of (arb), the knowledge base version of (Arb), on the basis that it is not consistent with (KP4) and (KP6). Our position is supported, firstly, by the fact that (Arb) is an appropriate formalisation of the intuition of arbitration, while we have shown that (KP4) is suspect as a property to be applied universally. And secondly, the arbitration operations $\Delta_{\min}$ and $\Delta_{\max}$ that we have presented seem to be valid both as combination operations in general and arbitration operations in particular.

The results also provide support for the use of (Maj) as a suitable postulate for the subclass of majority operations. It is compatible with the view of majority espoused by Konieczny and Pino-Pérez, since (Maj) is a generalisation of (maj); their knowledge base version of a majority postulate.

And finally, section 4.5 provides evidence that combination operations need not be commutative, i.e. they need not satisfy (Comm). Instead, (Comm) should be seen as picking out an interesting subclass of combination operations which are worth studying on their own.

In conclusion, then, the main contributions of this paper are to argue that it is useful to consider combination operations in general, and merging, arbitration and majority operations, in particular, on the level of epistemic states, and to provide an initial contribution towards the construction of a general framework for combination operations. An immediate point of departure for further research is the investigation of links between combination operations in this context and the huge body of related (but not identical) work in social choice theory [Arr63]. In particular, the question of devising strategy proof voting procedures seems to be very relevant for combination operations defined on epistemic states. We are currently investigating the problem of strategy-proofness in this context. And finally, on a more practical level, the question of combining lists of infobases is one that needs to be investigated as well.

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