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SYSTEMATIC WITHDRAWAL

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ABSTRACT. Although AGM theory contraction (Alchourrón et al., 1985; Alchourrón and Makinson, 1985) occupies a central position in the literature on belief change, there is one aspect about it that has created a fair amount of controversy. It involves the inclusion of the postulate known as Recovery. As a result, a number of alternatives to AGM theory contraction have been proposed that do not always satisfy the Recovery postulate (Levi, 1991, 1998; Hansson and Olsson, 1995; Fermé, 1998; Fermé and Rodriguez, 1998; Rott and Pagnucco, 1999). In this paper we present a new addition, *systematic withdrawal*, to the family of withdrawal operations, as they have become known. We define systematic withdrawal semantically, in terms of a set of preorders, and show that it can be characterised by a set of postulates. In a comparison of withdrawal operations we show that AGM contraction, systematic withdrawal and the *severe withdrawal* of Rott and Pagnucco (1999) are intimately connected by virtue of their definition in terms of sets of preorders. In a future paper it will be shown that this connection can be extended to include the epistemic entrenchment orderings of Gärdenfors (1988) and Gärdenfors and Makinson (1988) and the refined entrenchment orderings of Meyer et al. (2000).

KEY WORDS: belief change, belief revision, severe withdrawal, theory contraction, withdrawal

1. INTRODUCTION

Although AGM theory contraction (Alchourrón et al., 1985; Alchourrón and Makinson, 1985) occupies a central position in the literature on belief change, there is one aspect about it that has created a fair amount of controversy. It involves the inclusion of the postulate known as Recovery. The Recovery postulate is part of a formal expression of the principle of Informational Economy, the idea that an agent should try to keep the loss of information to a minimum. As a result, a number of alternatives to AGM theory contraction have been proposed that do not satisfy the Recovery postulate (Levi, 1991, 1998; Hansson and Olsson, 1995; Fermé, 1998; Fermé and Rodriguez, 1998; Rott and Pagnucco, 1999). Following a suggestion by Makinson (1987), belief removal operations satisfying all the basic AGM contraction postulates, except (possibly) for Recovery, have become known as *withdrawal* operations.

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In this paper we present a new addition, *systematic withdrawal*, to the family of withdrawal operations. We define systematic withdrawal semantically, in terms of a set of partial orders. Although the semantic definition is instrumental in explaining the motivation for introducing systematic withdrawal, we show that the latter is also characterised by a set of postulates. In a comparison of withdrawal operations, we show that AGM contraction, systematic withdrawal and the *severe withdrawal* of Rott and Pagnucco (1999) are intimately connected by virtue of their definition in terms of sets of preorders. These semantic constructions are then used to show that AGM contraction, systematic withdrawal and severe withdrawal are all interdefinable; indeed, interchangeable in the sense of Gärdenfors (1988). We argue that the close connection between these forms of withdrawal can best be understood in terms of their semantic construction. In a future paper it will be shown that the connection between these forms of withdrawal can be extended to include the epistemic entrenchment orderings of Gärdenfors (1988) and Gärdenfors and Makinson (1988) and the refined entrenchment orderings of Meyer et al. (2000).

1.1. *Preliminaries*

For the rest of this paper L denotes a propositional language, closed under the usual propositional connectives, and containing the symbols $\top$ and $\bot$. We assume L to have a two-valued model-theoretic semantics defining truth and falsity. The set of interpretations of L is denoted by U. We use $\Vdash$ for the relation from U to L denoting satisfaction and we assume that $\Vdash$ behaves classically with respect to the propositional connectives. It thus follows that for every $u \in U$ and every $\alpha \in L$, $u \Vdash \alpha$ or $u \Vdash \neg\alpha$. We use $\top$ and $\bot$ as canonical representatives for the logically valid and logically invalid wffs respectively. For concreteness the reader may think of the logic under consideration as a (possibly infinitely generated) propositional logic. For every $X \subseteq L$, $M(X) = \{x \in U \mid x \Vdash \alpha \text{ for every } \alpha \in X\}$ is the set of *models* of X, and for $\alpha \in L$ we write $M(\alpha)$ instead of $M(\{\alpha\})$. Entailment (from $\wp L$ to L) for L is defined as follows: $X \vDash \beta$ iff $M(X) \subseteq M(\beta)$, and for $\alpha, \beta \in L$ we write $\alpha \vDash \beta$ instead of $\{\alpha\} \vDash \beta$. We also require of $\vDash$ to satisfy *compactness*. That is, for every $X \subseteq L$ and every $\alpha \in L$, $X \vDash \alpha$ iff $X_F \vDash \alpha$ for some finite subset X_F of X. By $\alpha \equiv \beta$ we understand that α and β are logically equivalent, i.e. $\alpha \vDash \beta$ and $\beta \vDash \alpha$. Closure under entailment is denoted by Cn. A *theory* or a *belief set* is a set $K \subseteq L$ closed under entailment. A set $X \subseteq L$ is *satisfiable* iff $M(X) \neq \emptyset$, iff $Cn(X) \neq L$. For every $V \subseteq U$, the *theory determined by* V is $Th(V) = \{\alpha \in L \mid x \Vdash \alpha \text{ for every } x \in V\}$, and for $x \in U$ we write $Th(x)$ instead of $Th(\{x\})$. A set $X \subseteq L$ *axiomatises* a set of interpretations V iff $Cn(X) = Th(V)$. For a set $X \subseteq L$, the *expansion* of X by a wff $\alpha \in L$ is defined as $X + \alpha = Cn(X \cup \{\alpha\})$.

A preorder $\sqsubseteq$ (i.e. a reflexive and transitive binary relation) on a set X that is also connected is called a *total preorder*. For any preorder $\sqsubseteq$ on a set X, we write $x \sqsubset y$ iff $x \sqsubseteq y$ and $y \not\sqsubseteq x$, $x \equiv_{\sqsubseteq} y$ iff $x \sqsubseteq y$ and $y \sqsubseteq x$, $x \parallel_{\sqsubseteq} y$ iff $x \not\sqsubseteq y$ and $y \not\sqsubseteq x$, and we let $[x]_{\sqsubseteq} = \{y \mid x \equiv_{\sqsubseteq} y\}$.

Our examples are phrased in propositional languages, containing the usual propositional connectives, that are generated by at most three atoms. We use the letters p, q and r to denote these atoms, and interpretations of the languages will be represented by appropriate sequences of 0s and 1s, 0 representing falsity and 1 representing truth. The convention is that the first digit in the sequence represents the truth value of p, the second one the truth value of q and the third one the truth value of r.

And finally, we need the following two model-theoretic results, which are stated without proof.

LEMMA 1.1. *Suppose that K is a belief set and that $W \subseteq M(\alpha)$. Then*

$$(M(Th(M(K) \cup W)) \setminus M(K)) \subseteq M(\alpha).$$

LEMMA 1.2. *Let K be a belief set, and suppose that $X \subseteq M(\neg\alpha)$ and $W \subseteq M(\alpha)$. Then $M(Th(M(K) \cup X \cup W)) \cap M(\neg\alpha) = M(Th(M(K) \cup X)) \cap M(\neg\alpha)$.*

We conclude this section with some remarks about notation. During the course of the paper we shall be referring to various forms of belief removal generated from different types of orderings on U. Throughout the paper we use the following conventions (almost without exception). The symbols $\sim$ and $\approx$ denote generic belief removal operations, $-$ denotes AGM contraction (cf. Definition 2.1), $\div$ denotes systematic withdrawal (cf. Definition 5.2), and $\ddot{-}$ denotes severe withdrawal (cf. Definition 4.2). We use $\preceq$ to denote total preorders (reflexive, transitive and connected relations), $\leq$ to denote modular weak partial orders (cf. Definition 5.1) and $\preccurlyeq$ to denote layered preorders (cf. Definition 6.1).

2. AGM THEORY CHANGE

AGM theory change represents the *epistemic state* of an agent as a belief set and is concerned with two kinds of changes: the removal of an existing belief from the current belief set of the agent, known as AGM contraction, and the addition of a belief that might be inconsistent with the current belief set, known as AGM revision. The eight postulates for AGM contraction are given below.

(K−1) $K - \alpha = Cn(K - \alpha)$.

(K−2) $K - \alpha \subseteq K$.

(K−3) If $\alpha \notin K$ then $K - \alpha = K$.

(K−4) If $\nvDash \alpha$ then $\alpha \notin K - \alpha$.

(K−5) If $\alpha \equiv \beta$ then $K - \alpha = K - \beta$.

(K−6) If $\alpha \in K$ then $K \subseteq (K - \alpha) + \alpha$.

(K−7) $(K - \alpha) \cap (K - \beta) \subseteq K - (\alpha \wedge \beta)$.

(K−8) If $\beta \notin K - (\alpha \wedge \beta)$ then $K - (\alpha \wedge \beta) \subseteq K - \beta$.

(K−6) is the postulate known as Recovery.[1]

DEFINITION 2.1. A removal is a *basic AGM contraction* iff it satisfies (K−1) to (K−6). A removal is an *AGM contraction* iff it satisfies (K−1) to (K−8).

The eight postulates for AGM revision follow a similar pattern and are given below.

(K∗1) $K * \alpha = Cn(K * \alpha)$.

(K∗2) $\alpha \in K * \alpha$.

(K∗3) $K * \alpha \subseteq K + \alpha$.

(K∗4) If $\neg\alpha \notin K$ then $K + \alpha \subseteq K * \alpha$.

(K∗5) If $\alpha \equiv \beta$ then $K * \alpha = K * \beta$.

(K∗6) $\bot \in K * \alpha$ iff $\vDash \neg\alpha$.

(K∗7) $K * (\alpha \wedge \beta) \subseteq (K * \alpha) + \beta$.

(K∗8) If $\neg\beta \notin K * \alpha$ then $(K * \alpha) + \beta \subseteq K * (\alpha \wedge \beta)$.

DEFINITION 2.2. A revision is a *basic AGM revision* iff it satisfies (K∗1) to (K∗6). A removal is an *AGM revision* iff it satisfies (K∗1) to (K∗8).

Results in Grove (1988), Katsuno and Mendelzon (1991) and Boutilier (1994) show that AGM theory change can be characterised by a set of *total* preorders on U.

DEFINITION 2.3. Let $\preceq$ be any preorder on U.

(1) $x \in V \subseteq U$ is $\preceq$-*minimal* in V iff for every $y \in V$, $y \not\prec x$.
(2) $V \subseteq U$ is $\preceq$-*smooth* iff for every $y \in V$ there is an $x \preceq y$ that is $\preceq$-minimal in V.
(3) $\preceq$ is smooth iff $M(\alpha)$ is $\preceq$-smooth for every α. We denote the set of $\preceq$-minimal elements of $M(\alpha)$ by $Min_{\preceq}(\alpha)$.
(4) Given an arbitrary set $X \subseteq L$, a preorder $\preceq$ on U is X-*faithful* iff $\preceq$ is smooth, $x \prec y$ for every $x \in M(X)$ and $y \notin M(X)$, and $x \not\prec y$ for every $x, y \in M(X)$.

The idea is to consider preorders in which the models of X, being the minimal, or "best" interpretations, are strictly below all other interpretations. These preorders should be seen as measures of the plausibility of interpretations; the lower down in the preorder, the more plausible an interpretation will be. The construction of AGM contraction and revision can then be described in terms of the following identities:

(Def $\sim$ from $\preceq$) $K \sim \alpha = Th(M(K) \cup Min_{\preceq}(\neg\alpha))$.

(Def $*$ from $\preceq$) $K * \alpha = Th(Min_{\preceq}(\alpha))$.

We state these results without proof in the theorem below, and use it throughout the rest of this paper without explicit references to it.

THEOREM 2.4. (1) *A removal is an AGM contraction iff it is obtained in terms of some K-faithful total preorder using* (*Def $\sim$ from $\preceq$*).

(2) *A revision is an AGM revision iff it is obtained in terms of some K-faithful total preorder using* (*Def $*$ from $\preceq$*).

Gärdenfors (1988) has shown that AGM contraction and AGM revision are interdefinable by courtesy of the two identities given below, respectively known as the *Levi identity* and the *Harper identity*.

(Def $*$ from $\sim$) $K * \alpha = (K \sim \neg\alpha) + \alpha$.

(Def $\sim$ from $*$) $K \sim \alpha = (K * \neg\alpha) \cap K$.

The semantic construction of these theory change operations can strengthen this result as follows:

DEFINITION 2.5. An AGM contraction $-$ and an AGM revision $*$ are *semantically related* iff they can be defined in terms of the same faithul total preorder using (Def $\sim$ from $\preceq$) and (Def $*$ from $\preceq$).

PROPOSITION 2.6. *Let $-$ be an AGM contraction and $*$ an AGM revision that are semantically related.*

(1) $-$ *can also be defined in terms of* $*$ *using* (*Def* $\sim$ *from* $*$).
(2) $*$ *can also be defined in terms of* $-$ *using* (*Def* $*$ *from* $\sim$).

Finally, two particular forms of AGM contraction deserve special mention. A *maxichoice contraction* is an AGM contraction for which, for every $\alpha \in K$, $K - \alpha$ can be described as $Th(M(K) \cup \{x\})$, where x is a countermodel of α. It is easily seen that maxichoice contractions are those AGM contractions that can be defined, using (Def $\sim$ from $\preceq$), in terms of the K-faithful total preorders that also happen to be linear when restricted to the countermodels of K. We refer to such orderings as *K-linear orders*. On the other side of the spectrum, a *full meet contraction* is an AGM contraction for which, for every $\alpha \in K$, $K - \alpha$ can be described as $Th(M(K) \cup M(\neg\alpha))$. It is easily verified that, for a fixed belief set K, the unique full meet contraction can be obtained, using (Def $\sim$ from $\preceq$), in terms of the (K-faithful) total preorder $\preceq$ defined as follows: $u \preceq v$ iff $u \in M(K)$ or $v \notin M(K)$.

3. To Recover or Not to Recover

At a first glance, the Recovery postulate (K–6) seems to be a reasonable requirement to impose on theory removal. It is a formalisation of the principle of *Informational Economy*:

(Informational Economy) Keep the loss of information to a minimum.

And while this is clearly a useful principle, it can have undesirable consequences if it is allowed to become the overriding concern. This is the background against which objections levelled at Recovery should be seen.

The Recovery postulate has been criticised by various authors, and for several different reasons (Makinson, 1987; Fuhrmann, 1991; Levi, 1991; Lindström and Rabinowicz, 1991; Niederee, 1991; Hansson, 1991, 1992, 1993, 1996). Arguably the most compelling of these arguments revolve around two counterexamples due to Hansson (1991, 1992), and also occurring in Hansson (1996, 1999). These counterexamples strongly suggest that concerns other than the retention of information should also play a role during the removal of beliefs. On the other hand, arguments in favour of Recovery, such as those of Nayak (1994) and Makinson (1997), are not so much arguments *for* its retention as they are arguments *against* its complete dismissal. It thus seems reasonable to investigate withdrawal operations that do not always satisfy Recovery, but that, nevertheless, retain its desirable features. Our concern shall be with removal operations that satisfy all the AGM contraction postulates, except, perhaps, for Recovery. We refer to such removals as principled withdrawals.

DEFINITION 3.1. A removal is a *principled withdrawal* iff it satisfies (K−1) to (K−5), (K−7) and (K−8).

To be able to justify such principled withdrawal operations, it is necessary to take a closer look at the intuition associated with AGM contraction. The inclusion of the Recovery postulate in the AGM framework is justified by an appeal to the principle of Informational Economy (Gärdenfors, 1988). If the principle of Informational Economy had been the overriding concern, it would have implied that the belief set resulting from an contraction of K by α should be a maximal subset of K that does not imply α. But this involves a restriction to the maxichoice contraction operations, which Alchourrón and Makinson (1982) have shown to be too strong for a general account of theory contraction.

Since AGM contraction is more than just maxichoice contraction, it follows that the principle of Informational Economy is not the only requirement in question, but rather one of several equally important guidelines. In particular, as Rott and Pagnucco (1999) argue in their excellent survey of withdrawal, the respective roles of the principles of Indifference and Preference in the construction of AGM contractions are as important as that of the principle of Informational Economy and that in defining AGM contraction, the principle of Informational Economy has, to some degree, already given way to the principle of Indifference.

(Indifference) Objects held in equal regard should be treated equally.

(Preference) Objects held in higher regard should be afforded a more favourable treatment.

It is our view that the use of some forms of withdrawal, such as severe withdrawal (cf. Section 4) and systematic withdrawal (cf. Section 5), can be justified by requiring that the role of the principle of Informational Economy be reviewed in order for both the principles of Indifference and Preference to take complete precedence over it. This, of course, is not to say that such forms of withdrawal are better or worse than AGM contraction. Rather, it is a matter of recognising that different initial assumptions lead to different outcomes. It should be possible, though, to compare different approaches that use the same basic assumptions. In Section 9 we provide a comparison of severe withdrawal and systematic withdrawal.

4. Severe Withdrawal

Rott and Pagnucco (1999) use the faithful total preorders to define the set of severe withdrawals.[2] Let us define the downset of a wff α in terms of a faithful preorder as follows:

(Def $\nabla_{\preceq}$) $\nabla_{\preceq}(\alpha) = \{x \mid \exists y \in Min_{\preceq}(\alpha), \text{ such that } x \preceq y\}$.

DEFINITION 4.1. For a faithful preorder $\preceq$ (which need not be total), we define $\nabla_{\preceq}(\alpha)$, the *downset* of a wff α in terms of $\preceq$ using (Def $\nabla_{\preceq}$).

The downset of α contains all the interpretations that are at least as low down in the ordering as the minimal models of α. Downsets are used to define severe withdrawal as follows:

(Def $\sim$ from $\nabla_{\preceq}$) $K \sim \alpha = Th(M(K) \cup \nabla_{\preceq}(\neg\alpha))$.

DEFINITION 4.2. A removal is a *severe withdrawal* iff it is defined in terms of a faithful total preorder using (Def $\sim$ from $\nabla_{\preceq}$).

Clearly, (Def $\sim$ from $\nabla_{\preceq}$) is an application, in terms of the faithful total preorders, of the principles of Indifference, Preference, and Informational Economy.

Rott and Pagnucco show that severe withdrawal is characterised by the following set of postulates.[3]

(K$\ddot{-}$1) $K \ddot{-} \alpha = Cn(K \ddot{-} \alpha)$.

(K$\ddot{-}$2) $K \ddot{-} \alpha \subseteq K$.

(K$\ddot{-}$3) If $\alpha \notin K$ then $K \ddot{-} \alpha = K$.

(K$\ddot{-}$4) If $\nvDash \alpha$ then $\alpha \notin K \ddot{-} \alpha$.

(K$\ddot{-}$5) If $\alpha \equiv \beta$ then $K \ddot{-} \alpha = K \ddot{-} \beta$.

(K$\ddot{-}$6) If $\vDash \alpha$ then $K \subseteq K \ddot{-} \alpha$.

(K$\ddot{-}$7) If $\nvDash \alpha$ then $K \ddot{-} \alpha \subseteq K \ddot{-} (\alpha \wedge \beta)$.

(K$\ddot{-}$8) If $\beta \notin K \ddot{-} (\alpha \wedge \beta)$ then $K \ddot{-} (\alpha \wedge \beta) \subseteq K \ddot{-} \beta$.

THEOREM 4.3 (Rott and Pagnucco, 1999). *A removal $\ddot{-}$ is a severe withdrawal iff it satisfies* (K$\ddot{-}$1) *to* (K$\ddot{-}$8).

The postulates for severe withdrawal differ from those for AGM contraction only on the sixth and seventh postulates. (K$\ddot{-}$6), which replaces Recovery, is the postulate sometimes referred to as Failure. (K$\ddot{-}$7) is a much stronger requirement than the corresponding AGM contraction postulate (K$-$7). It is a kind of monotonicity property, requiring that the removal of weaker wffs should always result in smaller belief sets. Rott and Pagnucco regard this as an intuitively plausible postulate which follows from the application of the principles of Indifference and Preference. In Section 9, we argue against the inclusion of this postulate, showing that it has some undesirable consequences, and that (K$\ddot{-}$7) is a consequence of the principles of Indifference and Preference only when they are applied to the faithful *total* preorders.

5. Systematic Withdrawal

In this section we introduce a set of principled withdrawals that are closely related to the severe withdrawals. Their construction is based on an application of the principles of Indifference, Preference and Informational Economy in a manner identical to that used in the construction of severe withdrawal. The only difference is that they are obtained using a set of faithful preorders other than the faithful *total* preorders.

DEFINITION 5.1. A weak partial order $\leq$ on a set X is *modular* iff for every $x, y, z \in X$, if $x \parallel_{\leq} y$ and $z < x$, then $z < y$. A transitive relation is *strictly modular* iff it is the strict counterpart of a modular weak partial order.

DEFINITION 5.2. A belief removal $\div$ is a *systematic withdrawal* iff it is defined in terms of a faithful modular weak partial order using (Def $\sim$ from $\nabla_{\leq}$).

The modular weak partial orders are the reflexive versions of the modular partial orders of Ginsberg (1986) and Lehmann and Magidor (1992), and was first introduced by Meyer et al. (2000). Informally, a modular weak partial order ensures that the elements of X are arranged in levels, with incomparable elements being regarded as on the same level. It is thus clear that the following two identities provide a natural connection between the total preorders (usually denoted by $\preceq$ and the modular weak partial orders (usually denoted by $\leq$).

(Def $\leq$ from $\preceq$) $\leq \; = \; \preceq \setminus \{(x, y) \in X \times X \mid x \neq y \text{ and } x \equiv_{\preceq} y\}$.

(Def $\preceq$ from $\leq$) $\preceq \,=\, \leq \cup \{(x, y) \in X \times X \mid x \parallel_{\leq} y\}$.

While the total preorders view all the elements on the same level as *equally plausible*, the modular weak partial orders regard all the elements on the same level as *incomparable*. This difference in intuition accounts for the differences between systematic and severe withdrawal, as discussed in Section 9.

The technical difference between systematic withdrawal and severe withdrawal lies in the difference between the downset of a wff α obtained from a total preorder and that obtained from a modular weak partial order. In the case where the downset is obtained from a modular weak partial order, the downset consists of the minimal models of α as well as all the interpretations strictly below them (which are all, of course, countermodels of α). In the case where the downset is obtained from a total preorder, all the interpretations mentioned above are included, as well as the countermodels of α on the same level as the minimal models of α. In Section 9 we shall see that this seemingly minor technical difference accounts for some fundamental differences between these two forms of principled withdrawal. For the moment, we provide a characterisation of systematic withdrawal in terms of a set of postulates.

(K÷1) $K \div \alpha = Cn(K \div \alpha)$.

(K÷2) $K \div \alpha \subseteq K$.

(K÷3) If $\alpha \notin K$ then $K \div \alpha = K$.

(K÷4) If $\nvDash \alpha$ then $\alpha \notin K \div \alpha$.

(K÷5) If $\alpha \equiv \beta$ then $K \div \alpha = K \div \beta$.

(K÷6) If $\vDash \alpha$ then $K \subseteq K \div \alpha$.

(K÷7) If $\gamma \in K \div (\alpha \wedge \gamma)$ then $\gamma \in K \div (\alpha \wedge \beta \wedge \gamma)$.

(K÷8) If $\beta \notin K \div (\alpha \wedge \beta)$ then $K \div (\alpha \wedge \beta) \subseteq K \div \beta$.

(K÷9) If $\alpha \in K$, $\alpha \vee \beta \in K \div \alpha$ and $\beta \notin K \div \alpha$ then $\alpha \in K \div (\alpha \wedge \beta)$.

(K÷10) If $\nvDash \alpha$ and $\beta \in K \div \alpha$ then $\alpha \notin K \div (\alpha \wedge \beta)$.

THEOREM 5.3. *A removal ÷ is a systematic withdrawal iff it satisfies* (K÷1) *to* (K÷10).

The first five postulates coincide with the first five AGM contraction postulates, and the first six coincide with the first six postulates for severe withdrawal. (K÷7) is a much weaker version of $(K\ddot{-}7)$. If a wff γ is entrenched enough in the belief set K so that it is retained when at least one of γ or α has to be discarded, then it should also be retained when at least one of γ or any wff logically stronger than α has to be discarded. (K÷8) is identical to (K−8) and $(K\ddot{-}8)$. (K÷9) and (K÷10) both introduce more restrictions on the relationship between withdrawals by different wffs. (K÷9) gives conditions under which a wff α should be retained and (K÷10) gives conditions under which α should be discarded.

6. Minimal-Equivalence

The definitions of severe withdrawal and systematic withdrawal have brought to the fore two important classes of preorders, the faithful total preorders and the faithful modular weak partial orders. In fact, both these classes are included in a larger class of faithful preorders, all with the same underlying intuition.

DEFINITION 6.1. A preorder $\preccurlyeq$ on a set V is called *layered* iff its strict counterpart is strictly modular.

Layered preorders appeal to the same intuition that underlies the total preorders and the modular weak partial orders. The idea is that the elements of V are arranged in levels, with elements in different layers being comparable. The difference between all these types of orderings concerns the way in which elements in the same layer are treated. So, while the total preorders regard all elements in the same layer as *comparable*, and the modular weak partial orders take all distinct elements in the same layer as *incomparable*, the layered preorders provide a compromise between these two extremes: they allow for both the comparability and the incomparability of elements in the same layer. The faithful layered preorders thus provide us with a degree of freedom that is lacking in both the faithful total preorders and the faithful modular weak partial orders. It allows us to regard some interpretations as being incomparable with respect to plausibility, and others to be equally plausible. Using this intuition, it is clear that every layered preorder is uniquely associated with a modular weak partial order and a total preorder. (And in fact, total preorders and modular weak partial orders *are* layered preorders.)

DEFINITION 6.2. A modular weak partial order $\leq$ on a set X, a total preorder $\preceq$ on X, and a layered preorder $\preccurlyeq$ on X are *semantically related* iff they have identical strict counterparts.

The removals obtained in terms of semantically related faithful layered preorders using (Def $\sim$ from $\preceq$) will be identical, due to a notion we refer to as minimal-equivalence.

DEFINITION 6.3. Two faithful preorders $\preceq$ and $\preceq'$ are *minimal-equivalent* iff $Th(Min_{\preceq}(\alpha)) = Th(Min_{\preceq'}(\alpha))$ for every $\alpha \in L$.

It is easily verified that a faithful layered preorder and its semantically related faithful total preorder and faithful modular weak partial order are minimal-equivalent.

PROPOSITION 6.4. *A removal and a revision defined in terms of a faithful layered preorder $\preccurlyeq$ using (Def $\sim$ from $\preceq$) and (Def $*$ from $\preceq$), can also be defined in terms of its semantically related faithful total preorder $\preceq$, and its semantically related faithful modular weak partial order $\leq$, using (Def $\sim$ from $\preceq$) and (Def $*$ from $\preceq$).*

Proof. Follows from the fact that $Min_{\precsim}(\alpha) = Min_{\preceq}(\alpha) = Min_{\leq}(\alpha)$ for every $\alpha \in L$. □

7. Revision-Equivalence

With the definition of severe withdrawal and systematic withdrawal, we now have, together with AGM contraction, three types of principled withdrawal at our disposal which, as it turns out, are very closely related. For a proper comparison of this relationship, it is instructive to commence with the description of a feature which Makinson (1987) refers to as revision-equivalence.

DEFINITION 7.1. Two withdrawals $\sim$ and $\approx$ are *revision-equivalent* iff $(K \sim \neg\alpha) + \alpha = (K \approx \neg\alpha) + \alpha$.

In other words, two withdrawals are revision-equivalent iff the revisions they define in terms of the Levi identity (Def $*$ from $\sim$), are identical. From Makinson (1987) we obtain the following results concerning the revision-equivalence of basic AGM contraction and (basic) withdrawal.

THEOREM 7.2. (1) *A revision-equivalent class of withdrawals contains a unique basic AGM contraction.*

(2) *The basic AGM contraction* $-$ *is the maximal element in the equivalence class* $[-]$ *of withdrawals that are revision-equivalent to* $-$. *That is, for every* $\sim$ *in* $[-]$, $K \sim \alpha \subseteq K - \alpha$ *for every* $\alpha \in L$.

To bring severe withdrawal into the picture, we need to restrict ourselves to the revision-equivalent classes which contain the AGM contractions.

DEFINITION 7.3. A revision-equivalent class is called *principled* iff it contains an AGM contraction.

Note that a withdrawal in a principled revision-equivalence class need not satisfy (K−7) and (K−8). A case in point is provided by Meyer (1999) (Example 6.2.11, p. 148).

Rott and Pagnucco (1999) provide the following results.

THEOREM 7.4. (1) *A principled revision-equivalent class contains a unique severe withdrawal.*

(2) *The severe withdrawal* $\ddot{-}$ *is the minimal element in the* (*principled*) *equivalence class* $[\ddot{-}]$ *of withdrawals that are revision-equivalent to* $\ddot{-}$ *and that satisfy* (K−8). *That is, for every* $\sim$ *in* $[\ddot{-}]$ *that satisfies* (K−8), $K \ddot{-} \alpha \subseteq K \sim \alpha$ *for every* $\alpha \in L$.[4]

It should come as no surprise that the revision-equivalence of an AGM contraction and a severe withdrawal is closely tied to their semantic definitions in terms of faithful total preorders.

DEFINITION 7.5. An AGM contraction and a severe withdrawal are *semantically related* iff they can be defined in terms of the same faithful total preorder using (Def $\sim$ from $\preceq$) and (Def $\sim$ from $\nabla_{\preceq}$) respectively.

THEOREM 7.6. *An AGM contraction and a severe withdrawal are semantically related iff they are revision-equivalent.*

Proof. Follows from Lemma 1.2 and the fact that every principled revision-equivalence class contains a unique AGM contraction and a unique severe withdrawal. □

It is easily established that the very notion of principled revision-equivalence hinges on the use of minimal-equivalent faithful layered preorders.

PROPOSITION 7.7. *Suppose* $\sim$ *and* $\approx$ *are two withdrawals which are in the same principled revision-equivalence class, and let* $*$ *be the AGM revision obtained in terms of* $\sim$ *and* $\approx$ *using the Levi identity,* (*Def* $*$ *from* $\sim$). *Furthermore, let* $\preceq$ *be any faithful layered preorder in terms of which*

$*$ is defined using (Def $*$ from $\preceq$). Then, for every $\alpha \in K \setminus Cn(\top)$, there is a $W_\alpha^{\sim} \subseteq M(\alpha)$ and a $W_\alpha^{\approx} \subseteq M(\alpha)$ such that

$$K \sim \alpha = Th(M(K) \cup Min_{\preceq}(\neg\alpha) \cup W_\alpha^{\sim})$$

and

$$K \approx \alpha = Th(M(K) \cup Min_{\preceq}(\neg\alpha) \cup W_\alpha^{\approx}).$$

Proof. Follows from Lemma 1.2. □

The significance of Proposition 7.7 is that it enables us to regard a set of minimal-equivalent faithful layered preorders as the basis for obtaining a principled revision-equivalent class of withdrawals.

It is easily established that some members of the principled revision-equivalent classes are in gross violation of the principles of Indifference, Preference and Informational Economy. Consider, for example, the smallest withdrawal $\overset{\cdots}{-}$ in a principled revision-equivalent class $[\overset{\cdots}{-}]$:

(Def $\overset{\cdots}{-}$ from $\preceq$) $K \overset{\cdots}{-} \alpha = \begin{cases} Th(M(K) \cup Min_{\preceq}(\neg\alpha) \cup M(\alpha)) \\ \quad \text{if } \alpha \in K \setminus Cn(\top), \\ K \text{ otherwise.} \end{cases}$

$\overset{\cdots}{-}$ adds *all* the models of $\neg\alpha$ to $M(K)$ during a withdrawal of α, regardless of how plausible (or implausible) they are. As such, it is not an appropriate candidate for principled withdrawal. It is most likely examples such as these which prompted Lindström and Rabinowicz (1991) to advance the thesis that any reasonable withdrawal should lie somewhere between AGM contraction and severe withdrawal. To be more precise, in a principled revision-equivalence class containing the AGM contraction $-$ and the severe withdrawal $\ddot{-}$, we should regard as reasonable, only those withdrawals $\sim$ for which $K \ddot{-} \alpha \subseteq K \sim \alpha \subseteq K - \alpha$ for every $\alpha \in L$. Following a suggestion by Rott (1992, 1995), we refer to this proposal as the *LR interpolation thesis*.

DEFINITION 7.8. A withdrawal is *reasonable* iff it satisfies the LR interpolation thesis.

The LR interpolation thesis requires a withdrawal by α to be effected by adding to $M(K)$, any subset of the models α that are at least as plausible as the most plausible models of $\neg\alpha$, together with these most plausible models of $\neg\alpha$. So, it does not guarantee an adherence to the principles of Preference and Indifference with regard to $M(\alpha)$. However, the LR interpolation thesis ensures the satisfaction of these two principles in terms of $M(\neg\alpha)$. That is, it will not add a model y of $\neg\alpha$ to $M(K)$ if there is a model x of $\neg\alpha$ which is more plausible than y. Furthermore, two models of $\neg\alpha$ that are equally plausible will always be treated similarly; they will either both be added to $M(K)$ or both not be added to $M(K)$. Also, the LR interpolation goes some way towards satisfying the principle of Preference when comparing models of α and models of $\neg\alpha$: Models of α that are less plausible than the most plausible models of $\neg\alpha$ are never added to $M(K)$ while the most plausible models of $\neg\alpha$ are always added.

We are now in a position to bring systematic withdrawal into the picture as well. It is perhaps to be expected that every principled revision-equivalence class contains a unique systematic withdrawal. And this is indeed the case, as the next proposition shows.

PROPOSITION 7.9. *Every principled revision-equivalence class contains a unique systematic withdrawal.*

It is easily seen that systematic withdrawal is also reasonable (that is, it satisfies the LR interpolation thesis).

PROPOSITION 7.10. *Every systematic withdrawal belongs to a principled revision-equivalence class, and is reasonable.*

In the context of revision-equivalence, the relationship between AGM contraction, systemic withdrawal, severe withdrawal, and the faithful layered preorders defining these different forms of principled withdrawal, is summarised in the following corollary.

COROLLARY 7.11. *Consider a principled revision-equivalence class $\mathcal{R}$ of withdrawals.*

(1) *There is a minimal-equivalence class $\mathcal{M}$ of faithful layered preorders such that, for every faithful layered preorder $\preceq$ in $\mathcal{M}$ and every withdrawal $\sim$ in $\mathcal{R}$, $K \sim \alpha = Th(M(K) \cup Min_{\preceq}(\neg\alpha) \cup W_{\alpha}^{\sim})$, where $W_{\alpha}^{\sim} \subseteq M(\alpha)$.*
(2) *$\mathcal{R}$ contains a unique AGM contraction $-$, a unique systematic withdrawal $\div$ that is also reasonable, and a unique severe withdrawal $\ddot{-}$.*
(3) *For every withdrawal $\sim$ in $\mathcal{R}$, $K \sim \alpha \subseteq K - \alpha$ for every $\alpha \in L$.*
(4) *For every withdrawal $\sim$ in $\mathcal{R}$ which satisfies* (K$-$8)*, $K \ddot{-} \alpha \subseteq K \sim \alpha$ for every $\alpha \in L$.*
(5) *The AGM contraction $-$ can be defined in terms of every faithful layered preorder $\preceq$ in $\mathcal{M}$, using* (*Def* $\sim$ *from* $\preceq$).
(6) *The systematic withdrawal $\div$ can be defined in terms of every faithful modular weak partial order $\leq$ in $\mathcal{M}$, using* (*Def* $\sim$ *from* $\nabla_{\preceq}$).

(7) *The severe withdrawal* $\ddot{-}$ *can be defined in terms of every faithful total preorder* $\preceq$ *in* $\mathcal{M}$*, using* (*Def* $\sim$ *from* $\nabla_{\preceq}$).

Proof. Follows from Proposition 7.7, Theorems 7.2 and 7.4, Propositions 7.9, 7.10, and 6.4, Theorem 5.3, and Theorem 4.3. □

8. Reasonable Withdrawal

This section is devoted to an investigation of the relationship between various reasonable withdrawals, with particular emphasis on AGM contraction, systematic withdrawal and severe withdrawal. We have seen that AGM contraction and severe withdrawal both occupy special positions in the revision-equivalence classes. The former provides an upper bound for reasonable withdrawal, and the latter a lower bound. As a result both can be defined in terms of the remaining reasonable withdrawals. In particular, $-$ can be obtained from any revision-equivalent reasonable withdrawal $\sim$ as follows

(Def $-$ from $\sim$) $K - \alpha = K \cap ((K \sim \alpha) + \neg\alpha)$.

And $\ddot{-}$ can be obtained from any revision-equivalent reasonable withdrawal $\sim$ in one of two ways:[5]

(Def $\ddot{-}$ from $\sim$) $\beta \in K \ddot{-} \alpha$ iff $\begin{cases} \beta \in K \sim (\alpha \wedge \beta) \text{ if } \nvDash \alpha, \\ \beta \in K \text{ otherwise.} \end{cases}$

(Def $\ddot{-}$ from $\sim$ (v2)) $K \ddot{-} \alpha = \begin{cases} \bigcap \{K \sim (\alpha \wedge \beta) \mid \beta \in L\} \text{ if } \nvDash \alpha, \\ K \text{ otherwise.} \end{cases}$

PROPOSITION 8.1. *Let* $-$ *and* $\ddot{-}$ *be an AGM contraction and a severe withdrawal respectively, that are revision-equivalent. Suppose that* $\sim$ *is a reasonable withdrawal that is revision-equivalent to* $\ddot{-}$ (*and* $-$). *Then*

(1) $-$ *can be defined in terms of* $\sim$ *using* (*Def* $-$ *from* $\sim$),
(2) $\ddot{-}$ *can be defined in terms of* $\sim$ *using* (*Def* $\ddot{-}$ *from* $\sim$), *and*
(3) $\ddot{-}$ *can be defined in terms of* $\sim$ *using* (*Def* $\ddot{-}$ *from* $\sim$ (v2)).

Since systematic withdrawal is reasonable, it follows from Proposition 8.1 that every systematic withdrawal $\div$ defines a revision-equivalent AGM contraction $-$ using (Def $-$ from $\sim$), and a revision-equivalent severe withdrawal $\ddot{-}$ using (Def $\ddot{-}$ from $\sim$). Being reasonable, $\div$ lies somewhere between $-$ and $\ddot{-}$, so to speak. Nevertheless, it is possible to define systematic withdrawal in terms of both AGM contraction and severe withdrawal. In particular, $\div$ can be defined in terms of $-$ as follows:

$$\textbf{(Def} \div \textbf{from} -) \;\; \beta \in K \div \alpha \text{ iff } \begin{cases} \alpha \vee \beta \in K - (\alpha \wedge \beta) \\ \quad \text{and } \alpha \notin K - (\alpha \wedge \beta) \\ \quad \text{if } \nvDash \alpha,\ \nvDash \beta,\ \alpha \in K, \\ \beta \in K \text{ otherwise}. \end{cases}$$

And $\div$ can be defined in terms of $\ddot{-}$ as follows:

$$\textbf{(Def} \div \textbf{from} \ddot{-}) \;\; \beta \in K \div \alpha \text{ iff } \begin{cases} \alpha \vee \beta \in K \ddot{-} \alpha \text{ and } \alpha \notin K \ddot{-} \beta \\ \quad \text{if } \nvDash \alpha,\ \nvDash \beta \text{ and } \alpha \in K, \\ \beta \in K \text{ otherwise}. \end{cases}$$

DEFINITION 8.2. An AGM contraction $-$, a systematic withdrawal $\div$, and a severe withdrawal $\ddot{-}$ are *semantically related* iff there is a faithful total preorder $\preceq$ and a semantically related faithful modular weak partial order $\leq$ such that

(1) $-$ is defined in terms of $\preceq$ (and $\leq$) using (Def $\sim$ from $\preceq$),
(2) $\div$ is defined in terms of $\leq$ using (Def $\sim$ from $\nabla_{\leq}$), and
(3) $\ddot{-}$ is defined in terms of $\preceq$ using (Def $\sim$ from $\nabla_{\preceq}$).

PROPOSITION 8.3. *Let $-$ be an AGM contraction, let $\div$ be a systematic withdrawal, and let $\ddot{-}$ be a severe withdrawal. Suppose that $-$, $\div$ and $\ddot{-}$ are semantically related.*

(1) *$-$ and $\div$ can also be defined in terms of one another using* (*Def $-$ from $\sim$*) *and* (*Def $\div$ from $-$*).
(3) *$\ddot{-}$ and $\div$ can also be defined in terms of one another using* (*Def $\ddot{-}$ from $\sim$*) *and* (*Def $\div$ from $\ddot{-}$*).

9. SYSTEMATIC VERSUS SEVERE WITHDRAWAL

Systematic withdrawal and severe withdrawal are motivated by similar concerns. Indeed, they apply the principles of Indifference, Preference and Informational Economy in the same manner, and the method of construction used is identical; they differ only in the choice of faithful layered preorders to apply to (Def $\sim$ from $\nabla_{\preceq}$). As a result, they have many features in common. Firstly, both these forms of withdrawal are special cases of Cantwell's (1999) fallback-based withdrawal. Moreover, it is easily verified that systematic withdrawal and severe withdrawal satisfy (K$-$7), and that severe withdrawal, like systematic withdrawal, satisfies (K$\div$7) and (K$\div$10).

The close relationship between systematic and severe withdrawal raises the question of whether the two notions ever coincide. Part of the answer to this question is easy. Whenever a faithful layered preorder $\preceq$ is both a total preorder and a modular weak partial order, the severe withdrawal and the systematic withdrawal defined in terms of $\preceq$ using (Def $\sim$ from $\nabla_{\preceq}$) are, by definition, identical. It is easy to see that this is the case only when $\preceq$ is a K-linear order (see page 420). Furthermore, if a severe withdrawal cannot be defined in terms of a K-linear order using (Def $\sim$ from $\nabla_{\preceq}$), then it is not a systematic withdrawal, and vice versa; at least for the finitely generated propositional languages.

Notwithstanding the similarities between systematic and severe withdrawal, there are fundamental differences between them as well. We now come to a number of properties that are indicative of the major differences. The intuitive plausibility of all these properties can be related to the following simple example.[6]

EXAMPLE 9.1. While reading about Cleopatra, I have come across one source claiming that she had a son, and another claiming that she had a daughter. Now consider the following three situations.

1. If I attend a talk about the life and times of Cleopatra, and the speaker, an expert on the subject, says something which prompts me to retract the belief that Cleopatra had a son, it seems reasonable to retain the belief that she had a daughter.
2. Similarly, if the speaker leads me to retract the belief that Cleopatra had a daughter, I should retain the belief that she had a son.
3. Suppose that the speaker relates an incident which is specific enough to cast doubts on my belief that she had a son *and* a daughter, but is too vague to indicate whether she didn't have a son, didn't have a daughter, or perhaps, did not have any children at all. In these circumstances, intuition dictates that I should retain the belief that she had a child, without committing myself to a belief about it being a son or a daughter.
4. Finally, suppose the speaker says something which leads me to retract my belief that she had a child. Intuitively, I should end up with a belief state that is complete agnostic about having a child (and I should most certainly not subscribe to the belief that she either had no children or had both a son and daughter).[7]

To formalise this example, let L be a propositional language generated by the two atoms p and q. Let p denote the assertion that Cleopatra had a son, and q the assertion that she had a daughter. Then $K = Cn(p, q)$. The four different situations described above are then formalised as follows:

1. $K \sim p = Cn(q)$.
2. $K \sim q = Cn(p)$.

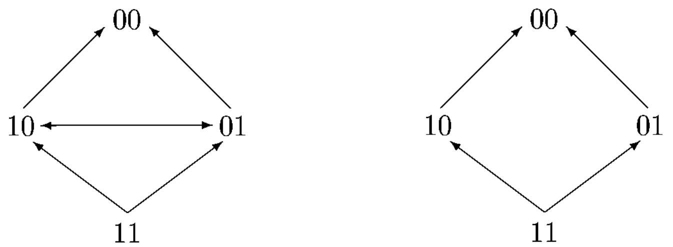

Figure 1. A graphical representation of a faithful total preorder $\preceq$ and the semantically related faithful modular weak partial order $\leq$. In both figures, two interpretations x and y are in the relevant faithful preorder iff (x, y) is in the reflexive transitive closure of the relation determined by the arrows.

3. $K \sim (p \wedge q) = Cn(p \vee q)$.
4. $K \sim (p \vee q) = Cn(\top)$.

It is easily verified that there is a systematic withdrawal which is able to accommodate all four parts of Example 9.1, but as we shall see below, the adherence to (K$\ddot{-}$7) ensures that severe withdrawal disallows the first three instances of this type of withdrawal. It is worth pointing out that severe withdrawal does allow the fourth instance of this type of withdrawal, and that AGM contraction allows the first three instances, but not the fourth one.

Let us now consider each of the relevant properties indicating the differences between systematic withdrawal and severe withdrawal. In doing so we shall be referring to the following example.

EXAMPLE 9.2. Let L be the propositional language generated by the two atoms p and q with $U = \{00, 01, 10, 11\}$. Furthermore, let $K = Cn(\{p, q\})$. Now, let $\preceq$ be the faithful total preorder defined as follows:

$$x \preceq y \quad \text{iff} \quad \begin{cases} y \in U \text{ if } x = 11, \\ y \in \{01, 10, 00\} \text{ if } x \in \{01, 10\}, \text{ and} \\ y = 00 \text{ if } x = 00, \end{cases}$$

and let $\leq$ be the semantically related faithful modular weak partial order (defined in terms of $\preceq$ using (Def $\leq$ from $\preceq$)). Figure 1 contains graphical representations of $\preceq$ and $\leq$. Let $\ddot{-}$ be the severe withdrawal defined in terms of $\preceq$ using (Def $\sim$ from $\nabla_{\preceq}$), and let $\div$ be the systematic withdrawal defined in terms of $\leq$ using (Def $\sim$ from $\nabla_{\leq}$). So $\nabla_{\preceq}(\neg(p \vee q)) = \nabla_{\leq}(\neg(p \vee q)) = U$ and thus $K \ddot{-} (p \vee q) = K \div (p \vee q) = Th(U) = Cn(\top)$. But $K \ddot{-} (p \vee q) + (p \vee q) = (K \div (p \vee q)) + (p \vee q) = Cn(p \vee q) \subset K$.

The first property to be considered is that expressed by (K$\ddot{-}$7). That it is not satisfied by systematic withdrawal, unlike severe withdrawal, is evident by considering the systematic withdrawal obtained in Example 9.2, and noting that $q \in K \div p$, but that $q \notin K \div (p \wedge q)$. Rott and Pagnucco (1999) argue in favour of (K$\ddot{-}$7) by making an appeal to the principles of Indifference and Preference. Observe that an $\alpha \wedge \beta$-withdrawal forces us to get rid of at least one of α or β. If α is given up, they argue, we can obtain an α-withdrawal by abandoning the same beliefs as when withdrawing $\alpha \wedge \beta$. And if β is given up, we might have to remove even more beliefs. This can be justified as follows. If α is given up during an $\alpha \wedge \beta$-withdrawal, the best models of $\neg(\alpha \wedge \beta)$ are at most as plausible as the best models of $\neg\alpha$. But the models of $\neg\alpha$ are also models of $\neg(\alpha \wedge \beta)$, and the best models of $\neg(\alpha \wedge \beta)$ can thus not be less plausible than the best models of $\neg\alpha$. From the principles of Indifference and Preference it then follows that a withdrawal by α should result in the addition of exactly the same interpretations as a withdrawal by $\alpha \wedge \beta$. On the other hand, if β is given up during a withdrawal by $\alpha \wedge \beta$, it follows by similar reasoning that the best models of $\neg(\alpha \wedge \beta)$ and of $\neg\beta$ are all equally plausible, with the best models of $\neg\alpha$ at most as plausible, and possibly less plausible. Consequently, the principles of Indifference and Preference dictate that a withdrawal by α should add at most as much interpretations as a withdrawal by $\alpha \wedge \beta$.

A careful analysis of the argument advanced above makes it clear that it relies heavily on the assumption that two interpretations can never be *incomparable*. In other words, it assumes the existence of a faithful total preorder to measure the relative plausibility of interpretations. But the moment this restriction is relaxed to, say, a faithful modular weak partial order, the postulate (K$\ddot{-}$7) is *not* sanctioned by the same appeal to the principles of Indifference and Preference. This can, perhaps, best be illustrated by Example 9.1. Even though both p and q are given up during a withdrawal by $p \wedge q$, we don't want either a withdrawal by p or a withdrawal by q to remove as much information as a withdrawal by $p \wedge q$.

The next property we consider is the one expressed by the postulate (K$\div$9). Unlike systematic withdrawal, it is not satisfied by severe withdrawal, a result which can be verified by noting that for the severe withdrawal $\ddot{-}$ in Example 9.2, $p \in K$, $p \vee q \in K \ddot{-} p$ and $q \notin K \ddot{-} p$, but $p \notin K \ddot{-} (p \wedge q)$. Intuitively, we can justify (K$\div$9) as follows. If $\alpha \vee \beta$, but not β, is retained after a withdrawal by α, it is an indication that β is more easily dislodged from K than α. Consequently, we should retain α, and discard β, when having to withdraw $\alpha \wedge \beta$.

Rott and Pagnucco (1999) show that severe withdrawal satisfies the following properties:

(Inclusion) Either $K \ddot{-} \alpha \subseteq K \ddot{-} \beta$ or $K \ddot{-} \beta \subseteq K \ddot{-} \alpha$.

(Decomposition) Either $K \ddot{-} (\alpha \wedge \beta) = K \ddot{-} \alpha$ or $K \ddot{-} (\alpha \wedge \beta) = K \ddot{-} \beta$.

(Converse conjunctive inclusion) If $\nvDash \alpha, \nvDash \beta$, and $K \ddot{-} (\alpha \wedge \beta) \subseteq K \ddot{-} \beta$ then $\beta \notin K \ddot{-} \alpha$.

(Expulsiveness) If $\nvDash \alpha$ and $\nvDash \beta$ then either $\alpha \notin K \ddot{-} \beta$ or $\beta \notin K \ddot{-} \alpha$.

Rott and Pagnucco regard it as regrettable that severe withdrawal satisfies Expulsiveness, in particular, and write as follows:

> Expulsiveness is an undesirable property since we do not necessarily want sentences that intuitively have nothing to do with one another to affect each other in belief contractions. This is the bitter pill we have to swallow if we want to adhere to the principles of Indifference and Preference.

We contend that it is the use of the faithful *total* preorders, and not these two principles that are the problem. This is made abundantly clear by noting that systematic withdrawal does not satisfy Expulsiveness. In fact, by considering the systematic withdrawal in Example 9.2, and taking p as α, and q as β in the four properties above, we see that systematic withdrawal doesn't satisfy any of the four properties above. Example 9.1 is thus evidence of the undesirability of these properties.

An analysis of the properties above creates the impression that, at least in some respects, severe withdrawal removes too much information from a belief set. This impression is strengthened by noting that severe withdrawal, unlike systematic withdrawal, includes the following particularly severe instance of withdrawal:

(Def $\dot{-}$) $K \dot{-} \alpha = \begin{cases} Cn(\top) \text{ if } \alpha \in K \setminus Cn(\top), \\ K \text{ otherwise.} \end{cases}$

PROPOSITION 9.3. *The belief removal $\dot{-}$ defined in (Def $\dot{-}$) is a severe withdrawal, but not a systematic withdrawal.*

Systematic withdrawal, on the other hand, retains a large amount of Recovery-like features, at least when compared to severe withdrawal. (K÷9), for example, implies the following weaker version of Recovery.

(Weak Recovery) If $\alpha \in K$ and $\alpha \notin K \div (\alpha \wedge \beta)$ then $\alpha \rightarrow \beta \in K \div \alpha$.

Now consider a variant of the scenario considered in Example 9.1, in which the reading of a book on Cleopatra convinced me that she had a son and a daughter ($p \wedge q$), and that this was the only reason for my belief that she had a child. So my current belief set can be represented as $K = Cn(p \wedge q)$. A subsequent discovery that the book was fictional then leads me to retract my belief that she had a son and a daughter. Moreover, it turns out that I also drop my belief that she had a child ($p \vee q$) in these circumstances (represented as $p \vee q \notin K \div p \wedge q$). Then (Weak Recovery) compels me to include $(p \vee q) \rightarrow (p \wedge q)$ in $K \div (p \vee q)$. That is, if I don't retract the belief that she had a son and a daughter, but instead retract the belief that she had a *child*, then subsequent historical evidence showing that she, in fact, had a child will force me to go back to my original position of believing that she had a son *and* a daughter.[8]

While this is indeed Recovery-like behaviour, our contention is that this is also intuitively justifiable behaviour. Here is why. The fact that $p \vee q$ is not in $K \div p \wedge q$ is a strong indication that p and q are not independent of each other. It should be viewed as a recognition that p and q were received simultaneously from a single source and should therefore be added or retracted simultaneously. So, if I retract $p \vee q$ only to add it again, the information that p and q stand or fall together should lead me to the conclusion that *both* p and q hold.

At the beginning of this section we saw that systematic withdrawal and severe withdrawal sometimes coincide. A related question is whether these two forms of withdrawal ever coincide with AGM contraction. It turns out that full meet contraction is the only case for which systematic withdrawal and AGM contraction are identical.

PROPOSITION 9.4. *Full meet contraction is the only AGM contraction that is a systematic withdrawal.*

With the exception of some cases involving a few trivial belief sets, though, severe withdrawal and AGM contraction always produce different results.

PROPOSITION 9.5. *Let K be such that for some $\alpha, \beta \in K$, $\nvDash \alpha$, $\nvDash \beta$ and $\alpha \not\equiv \beta$. Then severe withdrawal and AGM contraction never coincide.*

10. CONCLUSION

We have introduced a new form of withdrawal, systematic withdrawal, by means of a semantic construction involving the faithful modular weak partial orders on interpretations, as well as an axiomatic characterisation. The underlying intuition is the same as that associated with the severe withdrawals of Rott and Pagnucco (1999); the only difference being the type of preorders used in the semantic construction. It was shown that AGM contraction, severe withdrawal and systematic withdrawal are closely related by means of their respective semantic methods of construction.

Both the faithful total preorders, used in constructing severe withdrawals, and the faithful modular weak partial orders, used in constructing systematic withdrawals, can be seen as strict subsets of the general class of faithful layered preorders on interpretations. It is our contention that the class of withdrawals defined in terms of faithful layered deserves further study.

It is also possible to extend the known links between the epistemic entrenchment orderings of Gärdenfors (1988) and Gärdenfors and Makinson (1988), AGM contraction, and severe withdrawal, to include systematic withdrawal and the refined entrenchment orderings of Meyer et al. (2000). This will be discussed in detail in a future paper. The basis for obtaining these connections is provided by semantic construction methods in terms of faithful total preorders and faithful modular weak partial orders.

APPENDIX

PROPOSITION 2.6. *Let $-$ be an AGM contraction and $*$ an AGM revision that are semantically related.*

(1) *$-$ can also be defined in terms of $*$ using* (*Def* $\sim$ *from* $*$).
(2) *$*$ can also be defined in terms of $-$ using* (*Def* $*$ *from* $\sim$).

Proof. Let $\preceq$ be a faithful total preorder in terms of which $-$ and $*$ can be defined using (Def $\sim$ from $\preceq$) and (Def $*$ from $\preceq$). The proof of (1) is trivial and is omitted. For the proof of (2), it suffices to show that $Th(Min_{\preceq}(\alpha)) = Th(M(K) \cup Min_{\preceq}(\alpha)) + \alpha$. If $\neg\alpha \notin K$, it follows from the fact that $Min_{\preceq}(\alpha) \subseteq M(K)$. So let $\neg\alpha \in K$. Now observe that $M(Th(M(K) \cup Min_{\preceq}(\alpha)) + \alpha) = M(Th(M(K) \cup Min_{\preceq}(\alpha))) \cap M(\alpha)$. And by Lemma 1.1 it follows that $M(Th(M(K) \cup Min_{\preceq}(\alpha))) \cap M(\alpha) = M(Th(M(K) \cup Min_{\preceq}(\alpha))) \setminus M(K)$. So it suffices to show that $Th(Min_{\preceq}(\alpha)) = Th(M(Th(M(K) \cup Min_{\preceq}(\alpha))) \setminus M(K))$. The right-to-left inclusion follows from the fact that $Min_{\preceq}(\alpha) \subseteq M(Th(M(K) \cup Min_{\preceq}(\alpha))) \setminus M(K)$. For the left-to-right inclusion, pick a β such that $Min_{\preceq}(\alpha) \subseteq M(\beta)$. Then $\alpha \rightarrow \beta \in Th(M(K) \cup Min_{\preceq}(\alpha))$ and therefore $M(Th(M(K) \cup Min_{\preceq}(\alpha))) \setminus M(K) \subseteq M(\alpha \rightarrow \beta)$. But then, since $Min_{\preceq}(\alpha) \subseteq M(\alpha)$, it follows from Lemma 1.1 that $M(Th(M(K) \cup Min_{\preceq}(\alpha))) \setminus M(K) \subseteq M(\beta)$. □

LEMMA 0.1. *Systematic withdrawal satisfies* (K÷1)–(K÷10).

Proof. Let ÷ be a systematic withdrawal, and let $\leq$ be a faithful modular weak partial order from which ÷ is obtained using (Def $\sim$ from $\nabla_{\leq}$). (K÷1) is trivial. For (K÷2), observe that $M(K) \subseteq M(K) \cup \nabla_{\leq}(\neg\alpha)$. For (K÷3), suppose that $\alpha \notin K$. Then $\nabla_{\leq}(\neg\alpha) \subseteq M(K)$ and thus $K \div \alpha = K$. (K÷4) follows from the fact that $Min_{\leq}(\neg\alpha) \subseteq \nabla_{\leq}(\neg\alpha)$, while (K÷5) is trivial. For (K÷6), suppose that $\vDash \alpha$. Then $\nabla_{\preceq}(\neg\alpha) = \emptyset$ and thus $K \div \alpha = K$. For (K÷7), suppose that $\gamma \in K \div (\alpha \wedge \gamma)$. We only consider the case where $\nvDash \alpha$ and $\alpha \wedge \gamma \in K$. Then $\gamma \in K$ by (K÷2), $Min_{\leq}(\neg(\alpha \wedge \gamma)) \subseteq M(\neg\alpha) \cap M(\gamma)$ and $\nabla_{\leq}(\neg\alpha) \subseteq M(\gamma)$. Now pick any $x \in Min_{\leq}(\neg(\alpha \wedge \beta \wedge \gamma))$ and any $y \in Min_{\leq}(\neg(\alpha \wedge \gamma))$. (By smoothness, neither $Min_{\leq}(\neg(\alpha \wedge \beta \wedge \gamma))$ nor $Min_{\leq}(\neg(\alpha \wedge \gamma))$ is empty.) It is clear that $y \not< x$. So $\nabla_{\leq}(\neg(\alpha \wedge \beta \wedge \gamma)) \setminus Min_{\leq}(\neg(\alpha \wedge \beta \wedge \gamma)) \subseteq M(\gamma)$. To show that $\gamma \in K \div (\alpha \wedge \beta \wedge \gamma)$, it thus remains to show that $Min_{\leq}(\neg(\alpha \wedge \beta \wedge \gamma)) \subseteq M(\gamma)$. And if this were not the case, there would be a $z \in Min_{\leq}(\neg(\alpha \wedge \beta \wedge \gamma))$ such that $z \in M(\neg\gamma)$. But then $z \in Min_{\leq}(\neg(\alpha \wedge \gamma))$, thus contradicting $Min_{\leq}(\neg(\alpha \wedge \gamma)) \subseteq M(\neg\alpha) \cap M(\gamma)$.

For (K÷8), suppose that $\beta \notin K \div (\alpha \wedge \beta)$. We have to show that $K \div (\alpha \wedge \beta) \subseteq K \div \beta$. If $\alpha \wedge \beta \notin K$, then by (K÷3), $K \div (\alpha \wedge \beta) = K$, and thus also $K = K \div \beta$ (because

$\beta \notin K \div (\alpha \wedge \beta) = K$), from which the result follows. So we suppose that $\alpha \wedge \beta \in K$. Now, pick an $\alpha \in K \div (\alpha \wedge \beta)$. Then $M(K) \cup \nabla_{\leq}(\neg(\alpha \wedge \beta)) \subseteq M(\alpha)$ and so $M(K) \subseteq M(\alpha)$ and $\nabla_{\leq}(\neg(\alpha \wedge \beta)) \subseteq M(\alpha)$. We have to show that $M(K) \cup \nabla_{\leq}(\neg\beta) \subseteq M(\alpha)$. We already have that $M(K) \subseteq M(\alpha)$. To show that $\nabla(\leq \neg\beta) \subseteq M(\alpha)$, it suffices to show that $\nabla_{\leq}(\neg\beta) \subseteq \nabla_{\leq}(\neg(\alpha \wedge \beta))$. If we can show that $Min_{\leq}(\neg\beta) \subseteq Min_{\leq}(\neg(\alpha \wedge \beta))$, it immediately follows from (Def $\nabla_{\leq}$) that $\nabla_{\leq}(\neg\beta) \subseteq \nabla_{\leq}(\neg(\alpha \wedge \beta))$. So pick any $y \in Min_{\leq}(\neg\beta)$ and assume that $y \notin Min_{\leq}(\neg(\alpha \wedge \beta))$. Since $y \in M(\neg(\alpha \wedge \beta))$, it follows by the smoothness of $\leq$ that there is an $x \in Min_{\leq}(\neg(\alpha \wedge \beta))$ such that $x < y$. Because $y \in Min_{\leq}(\neg\beta)$, it must be the case that $x \in M(\neg\alpha \wedge \beta)$, and since $\leq$ is a modular weak partial order it then also follows that $Min_{\leq}(\neg(\alpha \wedge \beta)) \subseteq M(\beta)$. Moreover, since $y \in Min_{\leq}(\neg\beta)$ and since $x < y$ it has to be the case that for every $v \in Min_{\leq}(\neg(\alpha \wedge \beta))$ and every $u \leq v$, $u \in M(\beta)$. But then $\nabla_{\leq}(\neg(\alpha \wedge \beta)) \in M(\beta)$, contradicting the supposition that $\beta \notin K \div (\alpha \wedge \beta)$. For (K$\div$9), suppose that $\alpha \in K$, $\alpha \vee \beta \in K \div \alpha$ and $\beta \notin K \div \alpha$. We only consider the case where $\not\models \alpha$. Then $Min_{\leq}(\neg\alpha) \subseteq M(\beta)$, $\nabla_{\leq}(\neg\alpha) \setminus Min_{\leq}(\neg\alpha) \subseteq M(\alpha)$, and $\nabla_{\leq}(\neg\alpha) \setminus Min_{\leq}(\neg\alpha) \not\subseteq M(\beta)$. So $u < v$ for every $u \in Min_{\leq}(\neg(\alpha \wedge \beta))$ and every $v \in Min_{\leq}(\neg\alpha)$, and therefore $\nabla_{\leq}(\neg(\alpha \wedge \beta)) \subseteq M(\alpha)$, from which it follows that $\alpha \in K \div (\alpha \wedge \beta)$. For (K$\div$10), suppose that $\not\models \alpha$ and $\beta \in K \div \alpha$. Then $\nabla_{\leq}(\neg\alpha) \subseteq M(\beta)$. Therefore $Min_{\leq}(\neg\alpha) \subseteq Min_{\leq}(\neg(\alpha \wedge \beta))$ and thus $\alpha \notin K \div (\alpha \wedge \beta)$. □

LEMMA 0.2. *If* $\not\models \alpha$ *and* $\sim$ *is a removal that satisfies* (K$\div$1)*,* (K$\div$4)*,* (K$\div$5)*,* (K$\div$7) *and* (K$\div$8)*, then* $\{\beta \mid \beta \in K \sim (\alpha \wedge \beta)\} = \bigcap\{K \sim (\alpha \wedge \beta) \mid \beta \in L\}$.

Proof. Suppose $\beta \in K \sim (\alpha \wedge \beta)$. Now pick any γ. By (K$\div$7), $\beta \in K \sim (\alpha \wedge \gamma \wedge \beta)$, and by (K$\div$4), $\alpha \wedge \gamma \wedge \beta \notin K \sim (\alpha \wedge \gamma \wedge \beta)$. Therefore $\alpha \wedge \gamma \notin K \sim (\alpha \wedge \gamma \wedge \beta)$ by (K$\div$1), and so $K \sim (\alpha \wedge \gamma \wedge \beta) \subseteq K \sim (\alpha \wedge \gamma)$ by (K$\div$5) and (K$\div$8), from which it follows that $\beta \in K \sim (\alpha \wedge \gamma)$. So we have shown that $\{\beta \mid \beta \in K \sim (\alpha \wedge \beta)\} \subseteq \bigcap\{K \sim (\alpha \wedge \beta) \mid \beta \in L\}$. The converse is trivial. □

LEMMA 0.3. *If* $\sim$ *satisfies* (K$\div$1) *to* (K$\div$10)*, the withdrawal* $\ddot{-}$ *defined in terms of* $\sim$ *using* (*Def* $\ddot{-}$ *from* $\sim$) *is a severe withdrawal.*

Proof. (K$\ddot{-}$1) to (K$\ddot{-}$6) follow easily from Lemma 0.2, and (K$\ddot{-}$7) follows easily from (K$\div$7). For (K$\ddot{-}$8), suppose that $\beta \notin K \ddot{-} (\alpha \wedge \beta)$. If $\not\models \alpha \wedge \beta$ then $\beta \notin K \sim (\alpha \wedge \beta \wedge \beta) = K \sim (\alpha \wedge \beta)$. And if $\models \alpha \wedge \beta$ then $\beta \notin K$, and thus $\beta \notin K \sim (\alpha \wedge \beta)$ by (K$\div$2). So in either case, $K \sim (\alpha \wedge \beta) \subseteq K \sim \beta$ by (K$\div$8). We need to show that $K \ddot{-} (\alpha \wedge \beta) \subseteq K \ddot{-} \beta$. The case where $\models \alpha \wedge \beta$ is trivial, and so we suppose that $\not\models \alpha \wedge \beta$. We only consider the case where $\not\models \beta$. We need to show that $\{\gamma \mid \gamma \in K \sim (\alpha \wedge \beta \wedge \gamma)\} \subseteq \{\gamma \mid K \sim (\beta \wedge \gamma)\}$. Suppose that $\gamma \in K \sim (\alpha \wedge \beta \wedge \gamma)$. If $\beta \wedge \gamma \notin K \sim (\alpha \wedge \beta \wedge \gamma)$, then $K \sim (\alpha \wedge \beta \wedge \gamma) \subseteq K \sim (\beta \wedge \gamma)$ by (K$\div$8), and so $\gamma \in K \sim (\beta \wedge \gamma)$. So we consider the case where $\beta \wedge \gamma \in K \sim (\alpha \wedge \beta \wedge \gamma)$. Since $\gamma \in K \sim (\alpha \wedge \beta \wedge \gamma)$, it follows from (K$\div$4) that $\alpha \wedge \beta \notin K \sim (\alpha \wedge \beta \wedge \gamma)$, and then by (K$\div$8) that $K \sim (\alpha \wedge \beta \wedge \gamma) \subseteq K \sim (\alpha \wedge \beta)$. Because $K \sim (\alpha \wedge \beta) \subseteq K \sim \beta$ we then have that $\beta \wedge \gamma \in K \sim \beta$, and therefore $\beta \in K \sim \beta$, contradicting $\not\models \beta$ and (K$\div$4). □

LEMMA 0.4. *Let* $\sim$ *be a withdrawal satisfying* (K$\div$1) *to* (K$\div$10)*. Now define the removal* $\ddot{-}$ *in terms of* $\sim$ *using* (*Def* $\ddot{-}$ *from* $\sim$)*, and define the removal* $\div$ *in terms of* $\ddot{-}$ *using* (*Def* $\div$ *from* $\ddot{-}$)*. Then* $\sim$ *and* $\div$ *are identical.*

Proof. By combining (Def $\ddot{-}$ from $\sim$) and (Def $\div$ from $\ddot{-}$) it suffices to show that

$$\beta \in K \sim \alpha \quad \text{iff} \quad \begin{cases} \alpha \vee \beta \in K \sim (\alpha \wedge (\alpha \vee \beta)) \text{ and } \alpha \notin K \sim (\alpha \wedge \beta) \\ \quad \text{if } \not\models \alpha,\ \not\models \beta,\ \alpha \in K, \\ \beta \in K \text{ otherwise.} \end{cases}$$

We only consider the case where $\nvDash \alpha, \nvDash \beta$ and $\alpha \in K$. If $\beta \in K \sim \alpha$ then $\alpha \vee \beta \in K \sim \alpha = K \sim (\alpha \wedge (\alpha \vee \beta))$ by (K÷5), and $\alpha \notin K \sim (\alpha \wedge \beta)$ follows from (K÷10). Conversely, if $\alpha \vee \beta \in K \sim (\alpha \wedge (\alpha \vee \beta)) = K \sim \alpha$, and $\alpha \notin K \sim (\alpha \wedge \beta)$, then $\beta \in K \sim \alpha$ by (K÷9). □

THEOREM 5.3. *A removal ÷ is a systematic withdrawal iff it satisfies* (K÷1) *to* (K÷10).

Proof. The left-to-right direction follows from Lemma 0.1. For the converse, suppose that $\sim$ satisfies (K÷1) to (K÷10). Now define $\ddot{-}$ in terms of $\sim$ using (Def $\ddot{-}$ from $\sim$) on page 430. By Lemma 0.3, $\ddot{-}$ is a severe withdrawal. So there is a faithful total preorder $\preceq$ from which $\ddot{-}$ can be obtained using (Def $\sim$ from $\nabla_{\preceq}$). Let $\leq$ be the faithful modular weak partial order which is semantically related to $\preceq$. By Proposition 8.3 on page 431, the systematic withdrawal ÷ obtained from $\leq$ using (Def $\sim$ from $\nabla_{\leq}$) can also be defined in terms of $\ddot{-}$ using (Def ÷ from $\ddot{-}$) on page 431. And by Lemma 0.4, ÷ is identical to $\sim$. □

PROPOSITION 7.9. *Every principled revision-equivalence class contains a unique systematic withdrawal.*

Proof. Pick any principled revision-equivalence class. By Theorem 7.2, it contains a unique AGM contraction $-$ which, by Proposition 6.4, can be defined in terms of a faithful modular weak partial order $\leq$. By Lemma 1.2, the systematic withdrawal ÷, defined in terms of $\leq$ using (Def $\sim$ from $\nabla_{\leq}$), is revision-equivalent to $-$. Now assume there is a different systematic withdrawal $\sim$ in this revision-equivalence class. By Theorem 5.3, it can be defined in terms of a faithful modular weak partial order $\preceq$ using (Def $\sim$ from $\nabla_{\preceq}$), where $\preceq$ is *not* minimal-equivalent to $\leq$. And then $\preceq$ defines an AGM contraction $\check{-}$ in terms of (Def $\sim$ from $\preceq$) which, though revision-equivalent to $\sim$, differs from $-$. But this contradicts the uniqueness of $-$ in the given revision-equivalence class. □

PROPOSITION 7.10. *Every systematic withdrawal belongs to a principled revision-equivalence class, and is reasonable.*

Proof. Consider any systematic withdrawal ÷. By definition, there is a faithful modular weak partial order $\leq$ in terms of which ÷ is defined using (Def $\sim$ from $\nabla_{\leq}$). By Lemma 1.2, the AGM contraction defined in terms of $\leq$ using (Def $\sim$ from $\preceq$) is revision-equivalent to ÷, and it thus follows that ÷ belongs to a principled revision-equivalence class. Furthermore, from Theorem 7.2, $K \div \alpha \subseteq K - \alpha$ for every $\alpha \in L$.

Now consider the faithful total preorder $\preceq$ obtained in terms of $\leq$ using (Def $\preceq$ from $\leq$), and let $\ddot{-}$ be the severe withdrawal defined in terms of $\preceq$ using (Def $\sim$ from $\nabla_{\preceq}$). Then $K \ddot{-} \alpha \subseteq K \div \alpha$ for every $\alpha \in L$, and by Lemma 1.2, $\ddot{-}$ is revision-equivalent to ÷. So ÷ satisfies the LR-interpolation thesis; i.e. it is reasonable. □

PROPOSITION 8.1. *Let $-$ and $\ddot{-}$ be an AGM contraction and a severe withdrawal respectively, that are revision-equivalent. Suppose that $\sim$ is a reasonable withdrawal that is revision-equivalent to $\ddot{-}$* (*and* $-$). *Then*

(1) *$-$ can be defined in terms of $\sim$ using* (*Def $-$ from $\sim$*)*,*
(2) *$\ddot{-}$ can be defined in terms of $\sim$ using* (*Def $\ddot{-}$ from $\sim$*)*, and*
(3) *$\ddot{-}$ can be defined in terms of $\sim$ using* (*Def $\ddot{-}$ from $\sim$* (v2)).

Proof. Let $\preceq$ be a faithful total preorder in terms of which $-$ is defined using (Def $\sim$ from $\preceq$). By Corollary 7.11, $\ddot{-}$ can be defined in terms of $\preceq$ using (Def $\sim$ from $\nabla_{\preceq}$). Since $\sim$ is reasonable, and therefore revision-equivalent to $-$, there is, by Lemma 1.2, a $W_\alpha \subseteq M(\alpha)$ such that $K \sim \alpha = Th(M(K) \cup W_\alpha \cup Min_{\preceq}(\neg\alpha))$, for every $\alpha \in K \setminus Cn(\top)$. We only consider the cases where $\nvDash \alpha$.

(1) Follows from Lemma 1.2.

(2) If $\beta \notin K \sim (\alpha \wedge \beta)$ then $\beta \notin K \ddot{-} (\alpha \wedge \beta)$, since $\sim$ is reasonable and revision-equivalent to $\ddot{-}$. So there is a $y \in M(K) \cup \nabla_{\preceq}(\neg(\alpha \wedge \beta))$ such that $z \in M(\neg\beta)$, and therefore $y \in Min_{\preceq}(\neg(\alpha \wedge \beta))$. Therefore $x \not\prec y$ for every $x \in Min_{\preceq}(\neg\alpha)$, and thus $\beta \notin K \ddot{-} \alpha$. Conversely, if $\beta \notin K \ddot{-} \alpha$ then $y \in M(\neg\beta)$ for some $y \in M(K) \cup \nabla_{\preceq}(\neg\alpha)$, and there is thus an $x \in Min_{\preceq}(\neg(\alpha \wedge \beta))$ such that $x \in M(\neg\beta)$. Therefore $\beta \notin K \sim (\alpha \wedge \beta)$.
(3) If $\gamma \notin \bigcap\{K \sim (\alpha \wedge \beta) \mid \beta \in L\}$ then there is a $\beta \in L$ such that $\gamma \notin K \sim (\alpha \wedge \beta)$. And then $\gamma \notin K \ddot{-} (\alpha \wedge \beta)$, since $\sim$ is reasonable and revision-equivalent to $\ddot{-}$. So there is a $z \in M(K) \cup \nabla_{\preceq}(\neg(\alpha \wedge \beta))$ such that $M(\neg\gamma)$. But then $\gamma \notin K \ddot{-} \alpha$, since $y \not\prec x$ for every $y \in Min_{\preceq}(\neg\alpha)$ and every $x \in Min_{\preceq}(\neg(\alpha \wedge \beta))$. Conversely, if $\gamma \notin K \ddot{-} \alpha$ then $\gamma \notin K \ddot{-} (\alpha \wedge \gamma)$ by part (2), from which the required result follows. □

PROPOSITION 8.3. *Let $-$ be an AGM contraction, let $\div$ be a systematic withdrawal, and let $\ddot{-}$ be a severe withdrawal. Suppose that $-$, $\div$ and $\ddot{-}$ are semantically related.*

(1) *$-$ and $\div$ can also be defined in terms of one another using (Def $-$ from $\sim$) and (Def $\div$ from $-$).*
(2) *$\ddot{-}$ and $\div$ can also be defined in terms of one another using (Def $\ddot{-}$ from $\sim$) and (Def $\div$ from $\ddot{-}$).*

Proof. Let $\preceq$ be a faithful total preorder in terms of which $-$ and $\ddot{-}$ are defined using (Def $\sim$ from $\preceq$) and (Def $\sim$ from $\nabla_{\preceq}$) respectively, and let $\leq$ be the faithful modular weak partial order that is semantically related to $\preceq$. Since semantic relatedness implies revision-equivalence, it follows from Proposition 8.1 that $-$ can be defined in terms of $\div$ using (Def from $\sim$), and that $\ddot{-}$ can be defined in terms of $\div$ using (Def $\ddot{-}$ from $\sim$). For the remaining results we only consider the case where $\nvDash \alpha, \nvDash \beta$ and $\alpha \in K$.

(1) Suppose that $\beta \in K \div \alpha$. Then $\nabla_{\leq}(\neg\alpha) \subseteq M(\beta)$ and therefore $Min_{\preceq}(\neg(\alpha \wedge \beta)) \subseteq M(\alpha \vee \beta)$ and $Min_{\preceq}(\neg\alpha) = Min_{\preceq}(\neg(\alpha \wedge \beta))$. Therefore $\alpha \vee \beta \in K - (\alpha \wedge \beta)$ and $\alpha \notin K - (\alpha \wedge \beta)$. Conversely, suppose that $\alpha \vee \beta \in K - (\alpha \wedge \beta)$ and $\alpha \notin K - (\alpha \wedge \beta)$. So $Min_{\preceq}(\neg\alpha) \subseteq Min_{\preceq}(\neg(\alpha \wedge \beta))$ and thus $\nabla_{\leq}(\neg\alpha) \subseteq \nabla_{\leq}(\neg(\alpha \wedge \beta)) \subseteq M(\alpha \vee \beta)$, from which it follows that $\beta \in K \div \alpha$.

(2) Suppose that $\beta \in K \div \alpha$. Then $\nabla_{\leq}(\neg\alpha) \subseteq M(\beta)$, and so $\nabla_{\preceq}(\neg\alpha) \subseteq M(\alpha \vee \beta)$, and thus $\alpha \vee \beta \in K \ddot{-} \alpha$. Furthermore, $y \not\prec x$ for every $x \in Min_{\preceq}(\neg\alpha)$ and every $y \in Min_{\preceq}(\neg\beta)$, and so $\alpha \notin K \ddot{-} \beta$. Conversely, suppose that $\alpha \vee \beta \in K \ddot{-} \alpha$ and $\alpha \notin K \ddot{-} \beta$. Then $\nabla_{\preceq}(\neg\alpha) \subseteq M(\alpha \vee \beta)$, which means $Min_{\preceq}(\neg\alpha) \subseteq M(\beta)$. Furthermore, $\nabla_{\preceq}(\neg\beta) \nsubseteq M(\alpha)$, and so $y \not\prec x$ for every $x \in Min_{\preceq}(\neg\alpha)$ and every $y \in Min_{\preceq}(\neg\beta)$. Therefore $\nabla_{\leq}(\neg\alpha) \setminus Min_{\preceq}(\neg\alpha) \subseteq M(\beta)$ and thus $\beta \in K \div \alpha$. □

PROPOSITION 9.3. *The belief removal $\dot{-}$ defined in (Def $\dot{-}$) is a severe withdrawal, but not a systematic withdrawal.*

Proof. It is easily verified that $\dot{-}$ is defined in terms of the faithful total preorder $\preceq$ using (Def $\sim$ from $\nabla_{\preceq}$), where $\preceq$ is defined as follows:

$$x \preceq y \quad \text{iff} \quad \begin{cases} y \in U \text{ if } x \in M(K), \\ y \in U \setminus M(K), \text{ otherwise} \end{cases}$$

and $\dot{-}$ is thus a severe withdrawal. Now assume that $\dot{-}$ is also a systematic withdrawal. Clearly $\dot{-}$ is revision-equivalent to itself, and by Corollary 7.11 it then follows that there is no other systematic withdrawal that is revision-equivalent to $\dot{-}$. Now, let $\leq$ be the faithful modular weak partial order that is semantically related to $\preceq$. Since $\leq$ is minimal-equivalent to $\preceq$, it follows from Corollary 7.11 that the systematic withdrawal $\div$ defined in terms of $\leq$ using (Def $\sim$ from $\nabla_{\preceq}$) is revision-equivalent to $\dot{-}$, and it is easily verified that $\div$ is not equal to $\dot{-}$; a contradiction. □

PROPOSITION 9.4. *Full meet contraction is the only AGM contraction that is a systematic withdrawal.*

Proof. The full meet contraction $-$ can be defined in terms of the following faithful modular weak partial order using (Def $\sim$ from $\preceq$): $x \leq y$ iff $x = y$, or $x \in M(K)$ and $y \notin M(K)$. It is therefore, by definition, a systematic withdrawal. Next we show that $-$ is the only belief removal that is both an AGM contraction and a systematic withdrawal. Pick any systematic withdrawal $\div$ other than $-$. So $K - \alpha \neq K \div \alpha$ for some $\alpha \in K \setminus Cn(\top)$. If $K - \alpha \nsubseteq K \div \alpha$, then there is a $\beta \in K - \alpha$ (and thus $\beta \in K$) such that $\beta \notin K \div \alpha$. Since $K - \alpha = Th(M(K) \cup M(\neg\alpha))$, there is therefore an $x \in M(K \div \alpha)$ such that $x \in M(\alpha \wedge \neg\beta)$. And thus $\beta \notin K \div \alpha + \alpha$, which is a violation of Recovery. So suppose that $K - \alpha \subset K \div \alpha$. Now let $\preceq$ be a faithful modular weak partial order in terms of which $\div$ is defined using (Def $\sim$ from $\nabla_{\preceq}$). Since $K - \alpha = Th(M(K) \cup M(\neg\alpha))$, it follows from $K - \alpha \subset K \div \alpha$ that there is a $\beta \in Th(M(K) \cup \nabla_{\preceq}(\neg\alpha))$ such that $y \in M(\neg\beta)$ for some $y \in M(\neg\alpha)$. So $\nvDash \alpha \vee \beta$, and since $Min_{\preceq}(\neg\alpha) \subseteq \nabla_{\preceq}(\neg\alpha)$, $Min_{\preceq}(\neg\alpha) \cap Min_{\preceq}(\neg(\alpha \vee \beta)) = \emptyset$, which means that for every $u \in Min_{\preceq}(\neg\alpha)$ and every $v \in Min_{\preceq}(\neg(\alpha \vee \beta))$, $u \prec v$. By smoothness, $Min_{\preceq}(\neg\alpha) \neq \emptyset$, and there is thus an $x \in M(\neg\alpha \wedge \beta)$ such that $x \prec z$ for every $z \in Min_{\preceq}(\neg(\alpha \vee \beta))$. So $x \in \nabla_{\preceq}(\neg(\alpha \vee \beta))$ and thus $\alpha \notin K \div (\alpha \vee \beta) + (\alpha \vee \beta)$. So $\div$ does not satisfy Recovery, and is therefore not an AGM contraction. □

PROPOSITION 9.5. *Let K be such that for some $\alpha, \beta \in K$, $\nvDash \alpha$, $\nvDash \beta$ and $\alpha \not\equiv \beta$. Then severe withdrawal and AGM contraction never coincide.*

Proof. Pick any severe withdrawal $\ddot{-}$ and let $\preceq$ be a faithful total preorder in terms of which $\ddot{-}$ can be defined using (Def $\sim$ from $\nabla_{\preceq}$). If $Min_{\preceq}(\neg(\alpha \leftrightarrow \beta)) \subseteq M(\alpha)$ then $\nabla_{\preceq}(\neg(\alpha \vee \neg\beta)) \nsubseteq M(\beta)$ and so $\beta \notin K \ddot{-} (\alpha \vee \neg\beta) + (\alpha \vee \neg\beta)$, which is a violation of Recovery. The remaining two cases, where $Min_{\preceq}(\neg(\alpha \leftrightarrow \beta)) \subseteq M(\beta)$, and where $Min_{\preceq}(\neg(\alpha \leftrightarrow \beta)) \nsubseteq M(\alpha)$ and $Min_{\preceq}(\neg(\alpha \leftrightarrow \beta)) \nsubseteq M(\beta)$, are similar. □

NOTES

[1] Observe that we have renumbered some of the AGM postulates for contraction. In particular, the numbers for (K−5) and (K−6) have been swapped around. This was done to facilitate an easier comparison with the postulates for severe and systematic withdrawal.

[2] Actually, they use Grove's systems of spheres, but it is easily translated into our framework of faithful total preorders.

[3] Pagnucco (1996) originally gave a different characterisation of severe withdrawal.

[4] This is a result derived from Observation 7 in (Rott and Pagnucco, 1999).

[5] Since (Def $\ddot{-}$ from $\sim$) and (Def $\ddot{-}$ from $\sim$ (v2)) define the same severe withdrawal when applied to any reasonable withdrawal, any further results involving (Def $\ddot{-}$ from $\sim$) should be seen as results involving (Def $\ddot{-}$ from $\sim$ (v2)) as well.

[6] This is a variant of the Cleopatra example of Hansson (1999).

[7] This is the Cleopatra example of Hansson (1999).

[8] We thank an anonymous referee for pointing out that systematic withdrawal satisfies (Weak Recovery) and constructing this example.

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